ALGEBRA FOR TODAY

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ALGEBRA FOR TODAY

BY

WILLIAM BETZ

VICE-PRINCIPAL OF THE EAST HIGH SCHOOL AND SPECIALIST IN MATHEMATICS
FOR THE PUBLIC SCHOOLS OF ROCHESTER, NEW YORK

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PREFACE

The evident necessity of making elementary algebra more meaningful to young pupils, and of harmonizing the views of progressive thinkers in the field of mathematics with the insistent new demands in education, has been the chief reason for the preparation of this book. It is based on years of careful experimental teaching and of continuous testing in many classes, the only safe foundation for any attempt at a *real* educational readjustment.

A detailed study of important educational reports, of the authoritative views of outstanding specialists, and of typical classroom results shows very clearly that a desirable reorganization of elementary algebra involves at least three major demands: (1) the really significant and basic ideas of the subject should receive much greater emphasis; (2) problem-solving must become more than a mere incident; (3) the technique must be made simpler and more purposeful.

Needless to say, the new algebra program must be a judicious blending of the old and the new. A complete break with the past is neither desirable nor possible.

The desire, then, to stress more definitely the new ideals in secondary mathematics and to provide an adequate background for both interest and effort was responsible for the principal features of this book. These may be summarized as follows:

1. *The entire course is built around six central objectives,* as follows: (1) the language and the ideas of algebra;

(2) the formula; (3) the equation; (4) the graph; (5) fundamental principles and processes; and (6) problem-solving.

2. *These objectives have as their connecting bond the study of relationships.* Every chapter aims to make a contribution to this underlying theme.

3. The first seven chapters, which are of basic importance, aim to present in a very simple, concrete manner a compelling motivation for the *functional program of algebra.* Attention is called especially to the introductory paragraphs of these chapters, which constitute a continuous commentary on the purpose of algebra.

4. The remaining ten chapters, which are built on this concrete, organic foundation, give careful attention to *the development of a sound technique* and to a further growth in the study of relationships.

5. Each chapter represents a fairly complete "unit" of work, a plan that makes for greater definiteness and simplicity of teaching.

6. Material for *development lessons* is presented at crucial points. (See pages 3–5, 19–20, 65–69, 83–84, and so on.)

7. *The exercises are carefully graded.* Definite provision is made for informal oral work.

8. *Practice exercises and tests are liberally provided* wherever they seem most useful. The time limits suggested are based on the observed work of pupils of average ability.

9. *Problem-solving receives due emphasis throughout the course,* advancing from very simple "translation" exercises to the solution of typical verbal problems.

10. Each chapter contains a *summary of specific objectives.* A *general summary and review* is given at the end of the book.

It was found impossible to carry out this program without going beyond the customary allotment of pages. The increased bulk of the book is due very largely, however, to the inclusion of features which have helped to give meaning and life to the subject, such as motivating discussions, pictures, practice material, supplementary honor work, and tests.

It would be a mistake, therefore, to assume that the completion of this book involves an unusual time allowance. Actual experience in many classes has proved that this is not the case. On the contrary, under this plan of teaching the pupil advances rapidly without a sacrifice of either an adequate appreciation of the importance of the subject or a mastery of the essential skills.

Moreover, it may not be superfluous to add that the material included in this course fully meets the requirements of the various examining bodies.

In conclusion, the author wishes to express his deep appreciation to all those who have assisted him either directly or indirectly. Much help and inspiration came to the author through his professional work at Teachers College, Columbia University. The constructive comments of hundreds of teachers from all parts of the country, before whom he demonstrated and discussed the ideas incorporated in this book, contributed to the development of a clearer perspective and to the avoidance of one-sided tendencies. The completion of this enterprise is primarily due, however, to the splendid support given to it by the author's colleagues and by the educational authorities of Rochester, New York.

Above all, a personal word of gratitude is extended to Mr. H. Carlisle Taylor, of the East High School, Rochester, who assisted very loyally in the preparation of portions of the manuscript, in the construction of tests, and in the reading of the proof.

WILLIAM BETZ

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GENERAL SUMMARY AND REVIEW 1

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TO THE TEACHER

1. Assuming that 180 lessons are available in a one-year course in elementary algebra, the following schedule of lessons is suggested for classes of average ability:

FIRST SEMESTER

CHAPTER	NUMBER OF LESSONS
I	11
II	7
III	7
IV	10
V	6
VI	8
VII	9
VIII	15
IX	10
Margin	7
	90

SECOND SEMESTER

CHAPTER	NUMBER OF LESSONS
X	9
XI	11
XII	12
XIII	9
XIV	9
XV	14
XVI	11
XVII	8
Margin	7
	90

2. Most of the material of Chapter II is familiar to the pupils from their work in arithmetic. Hence a large part of the chapter is merely a *review* in a new setting. In presenting the geometric formulas, the use of simple wire models has been found very valuable. Such models may readily be improvised.

3. Teachers desiring a more limited program may omit sections 19, 24, and 30, in Chapters II and III.

4. It is also possible to pass directly from Chapter I to Chapter IV without a serious break in the sequence. In that case the treatment of the *formula* as outlined in Chapters II and III may be regarded as *supplementary* material.

5. As a rule it is not advisable to complete at the first reading all the exercises which are given under each topic. Instead, a sufficient number of exercises should be reserved for review purposes. This is true particularly of the groups of verbal problems which occur throughout the text.

6. As soon as possible, all assignments should be made cumulative, due attention being given constantly to processes, equations, and problems. A monthly checking list has been found of great service in securing a proper emphasis on all the important topics.

7. After each test it is desirable to obtain the *median* of the class. For diagnostic purposes it is, of course, very valuable to enter on a special record sheet the score of each pupil for every question of the test and also to determine the percentage of correctness for each question. These records often furnish significant suggestions for the prevention and correction of typical errors.

8. Throughout the course the goal should be *real understanding* and *mastery* rather than a merely imitative performance.

9. In all the manipulative processes *accuracy is of far greater importance than speed.* The attempt to standardize speed norms may easily destroy the educational value of the subject. The uncritical transfer of the "speed idea" from arithmetic to algebra has done an immense amount of harm.

10. In a normal class period it is advisable to devote five or ten minutes to a brief but energetic review, fifteen or twenty minutes to a discussion of the new work, and the remaining minutes to miscellaneous written exercises.

ALGEBRA FOR TODAY

CHAPTER I

HOW LETTERS ARE USED IN ALGEBRA

1. The Shorthand of Algebra. In beginning the study of algebra, you naturally wish to know as soon as possible "what it is all about." It may help you to keep in mind that algebra is a kind of *shorthand,* by means of which it is possible to solve many mathematical problems more easily and quickly.

You have no doubt heard of *commercial shorthand,* which pupils study in business courses. They have to practice many hours a day, for months, before they can write down rapidly an ordinary conversation. Observe how a letter such as the one which is shown on the following page would appear when written in ordinary shorthand.

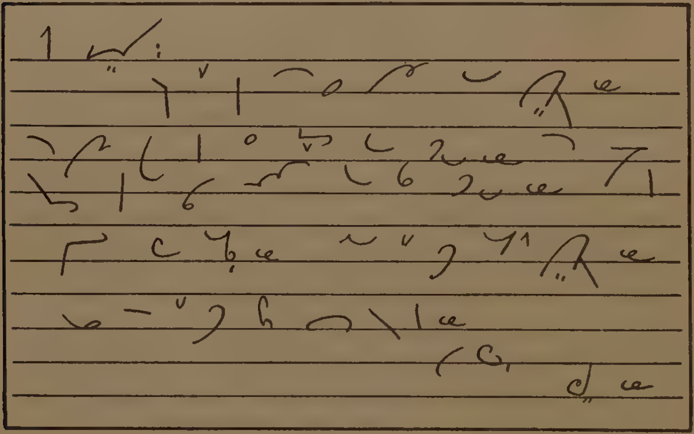
Unlike *commercial shorthand,* algebra uses the letters of the alphabet and certain symbols for the purposes of *mathematical shorthand.* In fact, the method used in algebra is very similar to one that is already familiar to you. You have often seen such expressions as U.S.A., C.O.D., M.D. The letters which appear in these abbreviations represent the *initials* of certain words; that is, in each case an entire word is replaced by its *first letter.* Whenever you sign your name by writing merely your initials, you are really applying the method used very commonly in the shorthand of algebra.

Dear Henry:

Today I had my first lesson in algebra. We learned that it is a kind of shorthand. We actually began to use a little of this shorthand. It looked very interesting. I think I shall enjoy algebra. Next week I shall tell you more about it.

Your friend,
James

Here is the same letter written in shorthand.



Exercises. *Using Initials*

1. Write the name of your school, such as North High School. How would you refer to it by means of initials?

2. What expressions are represented by the following abbreviations?

(1) C.O.D.
(2) f.o.b.
(3) A.D.
(4) Y.M.C.A.
(5) A.M.
(6) P.M.

3. Mention other common abbreviations such as those appearing in Exercise 2.

2. The Language of Algebra. It may also be said that algebra is a *language*. "It says more, in fewer words, than any other language." What, then, are the things we are to talk about in the study of algebra?

The answer is that the language of algebra has to do with things that can be *counted* or *measured*, and with certain processes growing out of counting and measuring. These processes are used to an ever increasing extent by many practical people, for a large variety of purposes. And so there are various ways of introducing the shorthand of algebra. We shall begin by studying some simple applications which can be illustrated very easily in any ordinary schoolroom.

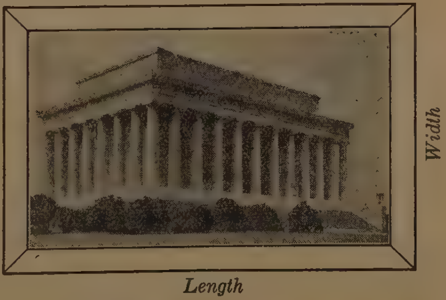
Exercises.* *Using the Shorthand of Algebra*

In the following exercises we shall try to learn how letters are used in algebra:

1. Select any rectangular object in the room, such as the frame of a picture. Draw a rectangle representing it.

Write the words *length* and *width* as shown in the figure.

EXPLANATION. You will recall that the length and the width of a rectangle are often called its *dimensions.*



2. If you were making the wooden frame of such a picture, you would have to know the required

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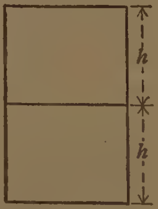
* It is suggested that these exercises be discussed informally. They constitute a development lesson.

dimensions. Only then could you cut off pieces of molding according to the length and the width desired. Similarly, whenever you *draw* a rectangular figure, you may not know the exact number of inches or feet in its dimensions, but you know that it has a *definite length* and a *definite width.*

Draw two lines representing the length and the width of the rectangle you drew in Exercise 1.

3. Since rectangles are used so frequently, it is often necessary to refer to their dimensions, even though we have not taken the trouble to find their numerical values. This is done briefly by using the shorthand of algebra. We replace the word *length* by its first letter, l, and the word *width* by w. Write these letters as shown below.

4. Each window of a schoolroom was found to have two equal panes of glass. What is the total height of these two panes of glass if the height of one is called h?



Solution. Evidently we can write:

Total height = h plus h. This may be shortened as follows:

$$t = h + h,$$

or

$$t = 2 \times h,$$

or

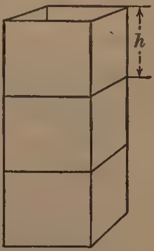
$$t = 2h.$$

This result does not tell us what the numerical value of the total height is, but simply that it is made up of two equal heights.

5. Following the method of Exercise 4, write in short form: $t = h + h + h$.

6. What is the meaning of $t = 4h$?

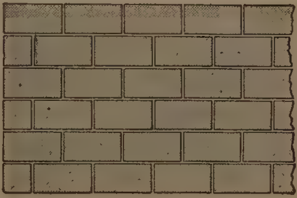
7. If three chalk boxes of the same height are placed on top of each other, and if the height of one box is called h, how would you express the total height of the three boxes?



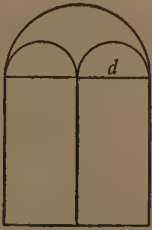
8. If ten such boxes were placed on top of each other, how would you denote their total height?

9. What is the meaning of $t = 20h$?

10. The floor of a schoolroom was found to consist of nineteen boards, all of the same width. If the width of one board is called w, what is the total width of the floor?



11. The adjoining diagram represents the face of a wall built of concrete blocks. The length of each block is l and its height is h. If the wall contains ten layers of these blocks, what is its height (ignoring thickness of mortar)? If one layer consists of forty blocks, what is the length of the wall?



12. If there are four windows in the schoolroom, of equal width, what is the combined width of these windows? (Use w.)

13. In the figure, if the diameter of the small semicircle is called d, what is the diameter of the large semicircle?

14. The altitudes of two mountains are denoted by a and $2a$. How do they compare in height?

15. If the length of the radius of a circle is denoted by r, how long is the diameter? Why?

3. Where Algebra is Used. Algebra is used in many trades, industries, and professions, for a variety of purposes. Thus, a business man has occasion to use algebra whenever he faces a hard problem in banking, in investment, or in insurance. The engineer needs a knowledge of algebra in designing and building machines, bridges, tunnels, roads, and so on. The architect and the contractor find algebra a necessary tool in solving difficult "construction" problems. A scientist uses algebra for many questions that arise in his laboratory. Each of these professions makes a different demand on algebra.

In the school it is impossible, however, to consider *technical* applications of algebra, because they would not be of interest to all, and because it would be difficult to explain them. For that reason only simple, everyday situations will be introduced in this course for the purpose of illustrating and applying the shorthand of algebra.

Oral Exercises. *Simple Applications of Algebra*

1. Express in brief form: $t = b + b + b + b$.

2. Shorten the statement $h = a + a + a + 2a$.

3. What is the meaning of $4b$? $5c$? $6d$?

4. If n stands for any *number*, what is the meaning of $2n$? $3n$? $\frac{1}{2}n$?

5. If w stands for the *weight* of a box, what is the meaning of $10w$? $50w$? $\frac{1}{5}w$?

6. If p stands for the *profit* of a certain firm, what is the meaning of $6p$? $\frac{1}{4}p$? $1.9p$?

7. If d means a certain *distance*, how long is $2d$? $5d$? $2.5d$?

8. In 1913 the *rent* for a certain house was r; today it is $3r$. Explain.

9. Last year the *cost* of a pound of sugar was c. Today it is $1.1c$. Explain.

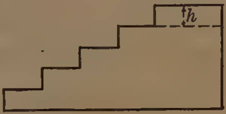
10. The *interest rate* of a bank during a certain month rose from r to $1.4r$. How much was the interest rate increased?

11. The noon *temperatures* of six cities on a certain day compared as follows: t; $1.1t$; $\frac{7}{8}t$; $1.2t$; $.85t$; $1.17t$. Explain.

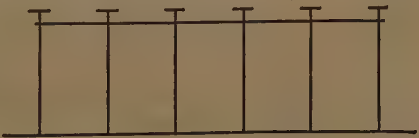
12. You have often heard of the *voltage* of an electric current. The voltages carried by three electric wires compared as follows: v; $4v$; $5.6v$. Explain.

13. The *volume* of air in a certain schoolroom was v. What is the combined volume of air of ten such rooms? The assembly hall had for its volume of air $20v$. Explain.

14. Each of these five stairs has the height h. How high is the staircase?



15. Telephone or telegraph poles usually are the same distance apart. If the distance between two successive poles, expressed in feet, is called f, how far apart are the first pole and the fourth pole? the fifth and the tenth?

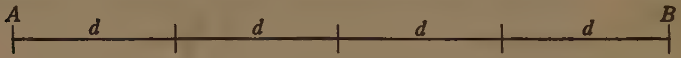


16. Mention the letters of the alphabet which have been used in the exercises above to abbreviate mathematical words. Can you extend the list? Give other familiar mathematical words which might be shortened in this way: (1) in business; (2) in building; (3) in engineering; (4) in everyday life.

4. Picturing Algebraic Shorthand. As you have seen, many letters of the alphabet may be used for the purposes of mathematical shorthand. The algebraic meaning of any particular letter depends on the situation in which it is used. The following exercises suggest how simple diagrams may help to make clear the meaning of algebraic shorthand.

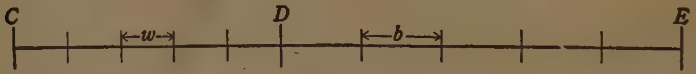
Exercises. *Picturing Algebraic Shorthand*

1. Suppose that a train takes four hours to go from Albany to Boston, covering a distance of d miles each hour.

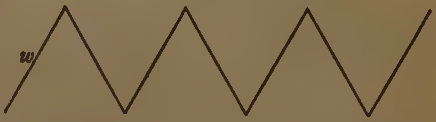


In the diagram, let A and B represent these two cities. Using the letter d, write the algebraic expression which represents the total distance from A to B. (The distance from A to B is often referred to as the distance AB.)

2. In the figure, how long is the distance CD? the distance DE?



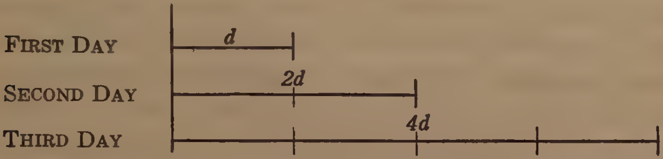
3. A wire was bent into a zigzag design, as shown, all the parts of the zigzag being equal. How long was the wire?



4. If w represents the width of a house, illustrate by a diagram the width $2w$ of another house.

5. If h represents the height of a tree, illustrate by a diagram the height of a tree which is $1\frac{1}{2}h$.

6. On an automobile trip Mr. Smith covered these distances on three successive days: d, $2d$, $4d$. The diagram shows how the distances may be pictured. Copy the figure, using any convenient length for d.

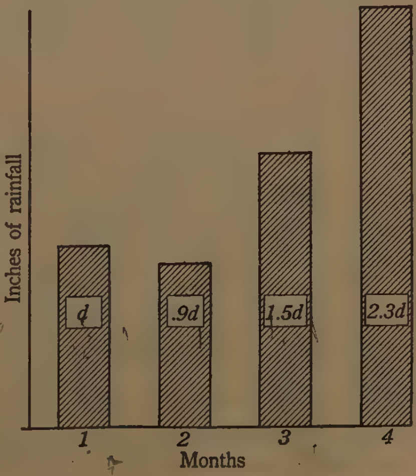


7. The lengths of four rivers are denoted by l, $2l$, $2.5l$, and $5l$. Represent their lengths by a diagram, using the method of Exercise 6.

8. The monthly profits of a certain business changed from June to December as follows: p; $1\frac{1}{2}p$; $2p$; $5p$; $4p$; $10p$. Represent p by any convenient line. Picture the monthly profits, using the method of Exercise 6.

9. Five business men invested money in the same enterprise. Their investments compared as follows: i; $2i$; $3\frac{1}{2}i$; $9i$; $10i$. Picture these investments.

10. The diagram shows the monthly amount of rainfall in inches at a certain place during four successive months. Can you interpret it? Compare the amount of rainfall during the first month with that during the third month.



5. Some New Words Explained. Like every other language, algebra has its own word list, or *vocabulary*. Most of this vocabulary was adopted many years ago. Read and study the explanations of new words very carefully. Much of your progress will depend on a clear grasp of their meaning. If you fail to give proper attention to the *vocabulary of algebra*, you will feel like a traveler in a strange land who has not learned the language of its people.

Algebraic Expression. Algebra, like arithmetic, treats of numbers. In algebra we also use letters to denote numbers. Any combination of numbers, letters denoting numbers, and signs of operation, when written according to the rules of algebra, is an *algebraic expression.*

Thus, $6h$, $h + h$, and $7w + 4$ are *algebraic expressions.*

able to
spell them
be able to
explain
them
give an
example

Factors. The expression $6w$ is a *product*, and 6 and w are its *factors*. In this expression 6 is a *numerical* factor and w is a *literal* factor. The factors of a product are the numbers which, when multiplied together, form the product.

Numerical Coefficient. In an expression such as $3w$, 3 is called the *numerical coefficient* of w, or simply the *coefficient* of w. A numerical coefficient shows how many times the letter following it is to be used as an addend.

Thus, $4w = w + w + w + w$.

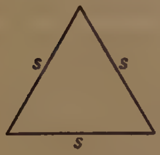
Remember that w has the same meaning as $1w$. Hence the coefficient of w is understood to be 1.

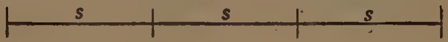
You will observe that in the exercises which you worked above, you constantly used algebraic expressions, factors, and coefficients. Test your knowledge of these terms by explaining, in your own words, the meaning of each.

6. Perimeter Formulas. If the three sides of a triangle are equal, the triangle is said to be *equilateral,* which means "equal-sided."

Place three equal pencils or rulers so as to form an equilateral triangle. Each of the rulers represents a side of the triangle. If the three rulers are placed end to end in a straight line, as shown in the figure below, their total length represents the boundary, or the *perimeter,* of the triangle. Let p stand for *perimeter.* Then we may write

$$p = s + s + s, \quad \text{or}$$
$$p = 3 \times s, \quad \text{or}$$
$$p = 3\,s.$$

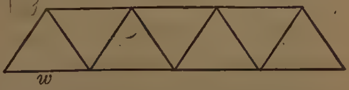




The statement $p = 3\,s$ is called the *perimeter formula* of the equilateral triangle. In the following exercises you will study the perimeter formulas suggested by figures which are in frequent use. It will help you to draw a sketch of each of these figures.

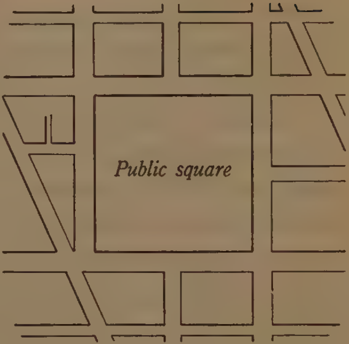
Exercises. *Perimeter Formulas*

1. If the side of an equilateral triangle is $2\,s$, write a formula for its perimeter (p).



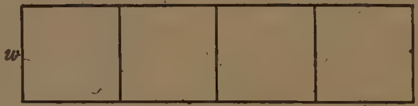
2. The figure represents a portion of a steel truss, often seen in bridges. Find the total length of the beams shown, if the length of each is w.

3. The perimeter of an equilateral triangle is $12\,d$. How long is each of its sides?



4. The public square of a certain town was to be inclosed by an iron fence. If the length of one side, in yards, is d, how many yards of fence had to be constructed? (Use p and d.)

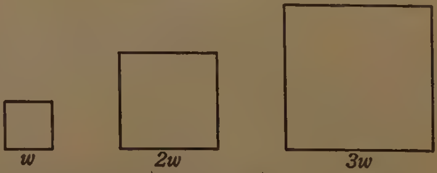
5. A picture frame is to consist of four square panels, as shown in the figure. If the width of the frame, in inches, is w, how many inches of molding must be used?



Solution. Let t represent the total length of the molding used. Then $t = 13w$. Explain.

6. What is the result, in Exercise 5, if the frame consists of five panels? twelve panels?

7. The wire models of three squares have the dimensions shown in the diagrams below. Find a formula for the total length of wire which is necessary to make the three models.



Solution.

$t = 4w + 8w + 12w$, or

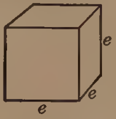
$t = 24w$. (Explain.)

The work may also be arranged *vertically*. In that case, to avoid confusion, the three perimeters are denoted by p_1, p_2, and p_3 respectively. Hence we may write

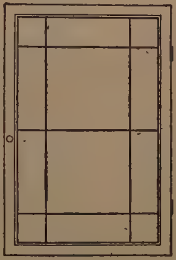
$$\begin{aligned} p_1 &= 4w \\ p_2 &= 8w \\ \underline{p_3} &\underline{= 12w} \\ t &= 24w \quad \text{(Explain.)} \end{aligned}$$

8. If the sides of four squares are w, $4w$, $6w$, and $8w$ respectively, find a formula for the total length (t) of their perimeters.

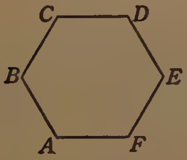
9. The adjoining picture represents a *cube*. Observe that a cube is bounded by six squares. If the length of one edge of the cube is denoted by e, what is the total length of all its edges? (Use t and e.)

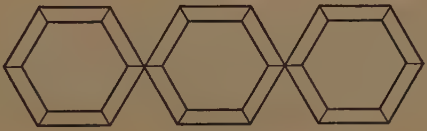


10. The frame of a rectangular window was divided by means of wooden partitions as shown in the figure. Denote a side of the small squares by w. The side of the large squares is three times as long. What is the total length of the wooden partitions?



11. The figure $ABCDEF$ below has six sides. It is called a *hexagon*. If all the sides are equal, the hexagon is said to be *equilateral*. Where have you seen a figure like this in use?





A parquet floor border consists of hexagonal units, as shown in the above figure. The pieces of wood making up the boundary of each hexagon are equal. If the longest dimension of each of these pieces is d, what is the total length of the pieces needed for 100 such hexagonal units?

12. The side of an equilateral hexagon is $3w$. Write a formula for its perimeter p.

13. What is a perimeter formula?

7. Similar Terms. You will now be able to understand the following explanations:

Term. Any expression such as $3w$, $8d$, or $7r$ is called a *term.*

Similar Terms. Two terms which have the same literal factor are said to be *similar.*

Thus, $5h$ and $6h$ are *similar terms.*

Dissimilar Terms. Two terms which do not have the same literal factor are said to be *dissimilar.*

Thus, $4a$ and $6b$ are *dissimilar terms.*

8. Combining Similar Terms. In each of the above exercises you had occasion to use similar terms. In making perimeter formulas such as those given above, it was necessary to *combine* similar terms. Thus, if the sides of two squares are denoted by w and $3w$, the total length of their perimeters is $4w + 12w$, or $16w$. The single term $16w$ is called the *sum* of the two original terms.

Moreover, in the case of these two squares, how much longer is the perimeter of the larger square? The answer is $12w - 4w$. This result may be expressed by the single term $8w$, which is called the *difference* of the two original terms.

From these exercises we may infer the following important

Rule for Combining Similar Terms

1. *The sum of similar terms* may be obtained by finding the *sum* of the coefficients of the given terms and multiplying it by the literal factor appearing in each of the terms.

Thus, $4a + 6a + 7a = 17a$.

2. *The difference of two similar terms* may be obtained by finding the *difference* between the coefficients of the given terms and multiplying it by the literal factor appearing in each of the terms.

Thus, $9a - 2a = 7a$.

Note. This rule will be called the *Rule of Similar Terms*, and the abbreviation S. T. may be used as a short way of referring to it.

Practice Exercises. *Combining Similar Terms*

In the following exercises combine the similar terms by applying the rule given above. Try to complete these exercises in 7 minutes. Write answers only.

1. Add: $\begin{array}{r} 4a \\ \underline{5a} \\ 9a \end{array} \qquad \begin{array}{r} 18c \\ \underline{20c} \end{array} \qquad \begin{array}{r} 100d \\ \underline{5d} \end{array} \qquad \begin{array}{r} 17n \\ \underline{n} \end{array} \qquad \begin{array}{r} w \\ \underline{41w} \end{array}$

2. Subtract: $\begin{array}{r} 20n \\ \underline{17n} \\ 3n \end{array} \qquad \begin{array}{r} 8a \\ \underline{5a} \end{array} \qquad \begin{array}{r} 7w \\ \underline{w} \end{array} \qquad \begin{array}{r} 20.4x \\ \underline{15.1x} \end{array} \qquad \begin{array}{r} 4\frac{1}{2}d \\ \underline{2\ d} \end{array}$

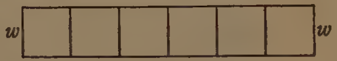
3. $5w + 6w = 11w$.
4. $10n + 8n + 6n = ?$
5. $8h + 7h + 4h = ?$
6. $2a + 4a + 10a = ?$
7. $5w + 6w + 9w = ?$
8. $40w - 30w = ?$
9. $10h + 15h + 6h = ?$
10. $90p + 50p - 40p = ?$
11. $\frac{1}{4}a + \frac{3}{4}a = ?$
12. $\frac{1}{3}w + \frac{2}{3}w = ?$
13. $4\frac{1}{2}x + 12x = ?$
14. $\frac{5}{6}d - \frac{1}{3}d = ?$
15. $9.5x - 7.2x = ?$
16. $4x + .62x = ?$
17. $20h - 1.72h = ?$
18. $12n + 10n - n = ?$
19. Given an equilateral triangle. One side is $4s$. Find the perimeter (p).

20. Given a square. One side is $10w$. Find p.

21. Given two squares. Their sides are s and $2s$ respectively. What is the total length of their perimeters? (Use t for total length of perimeters.)

22. Given an equilateral triangle and a square. The side of each is d. Find t.

23. A frame consists of six equal squares placed in a straight line. Find an expression for t.



24. If the perimeter of an equilateral triangle is $18w$, how long is one side?

Exercises. *Similar Terms*

In each of the following exercises apply the Rule of Similar Terms:

1. Add:

$20n$	$75b$	$91c$	$71x$	w
$30n$	$19b$	$17c$	x	$59w$

2. Subtract:

$30d$	$16w$	$19n$	$17.6x$	$7\frac{1}{2}s$
$19d$	$10w$	n	$8.9x$	$2\frac{1}{2}s$

3. Find the *sum* of the terms in each column:

$4a$	$3b$	$9w$	$20h$	$1.1d$
$6a$	$7b$	$10w$	$40h$	$2.3d$
$7a$	$9b$	w	$50h$	$4.9d$

4. Find the *sum* of the terms in each column:

$2n$	$17h$	$\frac{1}{3}n$	$7.5d$
$5n$	$15h$	$2\frac{1}{2}n$	$9.4d$
$6n$	$41h$	$5n$	$8.3d$
$7n$	$16h$	$9\frac{1}{3}n$	$5.7d$

5. Find the *difference* of the terms in each case:

$$\begin{array}{r} 90\,n \\ 80\,n \\ \hline \end{array} \qquad \begin{array}{r} 100\,d \\ 59\,d \\ \hline \end{array} \qquad \begin{array}{r} 7\frac{1}{2}\,c \\ 5\frac{1}{2}\,c \\ \hline \end{array} \qquad \begin{array}{r} .89\,e \\ .76\,e \\ \hline \end{array} \qquad \begin{array}{r} 207\,r \\ r \\ \hline \end{array}$$

6. $4\,a + 6\,a + 15\,a + 21\,a$.

7. $25\,n + 75\,n - 16\,n$.

8. $\frac{1}{2}\,d + \frac{1}{3}\,d$.

9. $2\frac{1}{4}\,n + 7\frac{1}{2}\,n$.

10. $.1\,b + 2.3\,b + 4.5\,b$.

11. $2\frac{1}{2}\,c + 3\frac{1}{3}\,c + 4\frac{1}{5}\,c$.

12. $8.2\,e + 9.3\,e + 14.6\,e$.

13. $9\,n + 7\frac{1}{2}\,n + 9\frac{1}{6}\,n - 10\frac{1}{2}\,n$.

14. $20\frac{1}{2}\,r + 50\frac{1}{4}\,r - 30\frac{1}{3}\,r$.

15. $7.2\,x + 9.2\,x - 3.6\,x$.

Observe how the Rule of Similar Terms is applied in the following arithmetical additions:

16.

$$\begin{array}{r} 5 \times 9 \\ 7 \times 9 \\ 4 \times 9 \\ \hline 16 \times 9 \\ = 144 \end{array} \qquad \begin{array}{r} \$700 \times .02 \\ \$900 \times .02 \\ \$600 \times .02 \\ \hline \$2200 \times .02 \\ = \$44 \end{array} \qquad \begin{array}{r} \$245 \times .06 \times \frac{1}{365} \\ \$654 \times .06 \times \frac{1}{365} \\ \$926 \times .06 \times \frac{1}{365} \\ \hline \$1825 \times .06 \times \frac{1}{365} \\ = \$0.30 \end{array}$$

17. Find the *sum* in each case:

$$\begin{array}{r} 90 \times .4 \\ 100 \times .4 \\ 20 \times .4 \\ \hline \end{array} \qquad \begin{array}{r} 8 \times \frac{1}{2} \\ 10 \times \frac{1}{2} \\ 12 \times \frac{1}{2} \\ \hline \end{array} \qquad \begin{array}{r} 33 \times \frac{4}{5} \\ 9 \times \frac{4}{5} \\ 8 \times \frac{4}{5} \\ \hline \end{array} \qquad \begin{array}{r} 16.4 \times 7.4 \\ 3.6 \times 7.4 \\ 12.0 \times 7.4 \\ \hline \end{array}$$

18. Combine $35 \times \frac{2}{3} - 14 \times \frac{2}{3}$.

19. A man deposits money in each of three banks, namely, \$100, \$200, and \$300 respectively. If each bank pays 4% interest, what is the total interest due after one year?

20. Given an equilateral triangle, a square, and an equilateral hexagon. Their sides are s, $2\,s$, and $3\,s$ respectively. Find the sum of the perimeters.

21. Given the wire models of four cubes. Their edges are of length e, $2\frac{1}{2}e$, $3\frac{1}{2}e$, and $4\frac{1}{4}e$ respectively. Find the total length of wire used for the four models.

22. The profits of a certain business, during six consecutive years, were p, $\frac{1}{2}p$, $3\frac{1}{2}p$, $8.1p$, $2.3p$, and $9.2p$ respectively. Find the total profit.

23. In the preceding exercise by how much do the profits of the last three years exceed those of the first three years?

24. The incomes of five men are i, $5i$, $4\frac{1}{3}i$, $6\frac{1}{2}i$, and $7i$ respectively. What is their total income? How much larger than the income of the first man is that of the fifth?

25. The distances traversed by an automobile, on five consecutive days, were d, $2d$, $1\frac{1}{2}d$, $2\frac{1}{3}d$, and $4d$. How does the total distance covered during the last three days compare with that of the first two days?

26. On a certain day a newsboy sold n papers. The next day he sold $2n$ papers. During the following four days his sales were $1\frac{1}{2}n$, $2\frac{1}{2}n$, $\frac{3}{4}n$, and $3n$ respectively. Picture each of these sales and also the total sales.

27. The weights of four trunks were found to be w, $3w$, $4\frac{1}{2}w$, and $1.4w$. What was the total weight of the trunks?

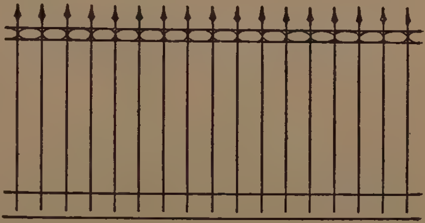
28. If the perimeter of a square is $8s$, how long is one side?

29. If the perimeter of an equilateral hexagon is $12w$, how long is one side?

30. A business man made five investments during a certain year. The sums invested were d, $2d$, $2.6d$, $3d$, and $4.2d$ respectively. Picture each of these investments and also the total investment.

9. Evaluation of a Formula. A very important process in the work with formulas is to be explained in the following paragraphs. Its meaning will become clear from some simple illustrations.

Example 1. In an iron fence there were 100 vertical rods of equal length. What is the total length of iron rod necessary for this fence?



If the length of one rod is called l, the total length of all these rods is expressed by the formula

$$t = 100\,l.$$

Now, how many feet of iron rod were actually used in this *particular* fence? The answer cannot be given until we know the exact length of *one* of the rods. Suppose that its length is 3 feet. Then the total length of the 100 rods, in feet, is evidently 100 × 3, or 300. To obtain this result, we *replace the letter* l, in the formula $t = 100\,l$, *by its numerical value.*

Similarly, if $l = 4$ feet, $t = 100 \times 4$, or 400 feet. In other words, *we find the value of* t *by substituting 4 for* l *in the given formula.* Observe that in each case *the value of* t *depends upon the value of* l.

Example 2. The pupils in a shop class were asked to make wire models of a square, of any convenient size.

The formula $p = 4\,w$ was readily seen to represent the perimeter of each of these squares. (Explain.) But before any particular pupil could tell how much wire he would need for his model, he had to decide how long one side was to be. Thus, the boy who made a 2-inch model required 4×2 (or 8) inches of wire. A 3-inch model required

4 × 3 (or 12) inches, and so on. In every case the actual length of the wire used in a particular model could be found by replacing w, in the formula $p = 4w$, by the exact length of a side.

The following *table* shows the dimensions chosen by six boys of the class, and the corresponding amount of wire used in each case. All the dimensions are given in inches.

	JOHN	HENRY	FRED	PAUL	ERNEST	ROBERT
Value of $w =$	3	5	6	$7\frac{1}{2}$	8.1	12
Value of $p = 4w =$	12	20	24	30	32.4	48

Such a table is often called a *substitution* table. Verify the numbers appearing in the second row of this table.

The illustrations given above will enable you to understand the following explanation:

Evaluation is the process of putting a number in place of a letter in a given algebraic expression, and thus finding the corresponding numerical value of the expression.

This process is often called *substitution*, for we are replacing a letter by a number. Observe that *the numerical value of an algebraic expression depends upon the numerical values which are substituted for the letters in the expression.*

Exercises. *Evaluating Formulas*

1. In the following table fill in the values of p by substituting in the formula the given values of w. All the dimensions are given in inches.

If $w =$	4	5	7	9	12
then $p = 8w =$	32				

2. Fill in the values of t by substituting in the formula the given values of d.

If $d =$	9	21	$4\frac{1}{2}$	$2\frac{1}{5}$	4.1
then $t = 10\,d =$	90				

3. Evaluate the formula $c = 8\,n$ for the given values of n.

$n =$	5	9	10	20	$25\frac{1}{2}$
$c =$	40				

4. Evaluate the formula $a = 9\,b$ for the given values of b.

$b =$	2	5	6.5	10.4	40.6
$a =$					

5. Sometimes a substitution table is arranged vertically. Supply in the second column the numerical values of t corresponding to the values given in the first column.

(1)

h	$t = 10\,h$
4	
7	
9.5	
$10\frac{1}{2}$	

$$\begin{array}{r} 6.5 \\ 9 \\ \hline 58.5 \end{array}$$

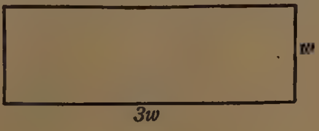
(2)

c	$t = 8\,c$
\$4.00	
\$5.20	
\$6.30	
\$7.90	

6. Write a formula for the perimeter of the adjoining rectangle. (Use p for perimeter.)

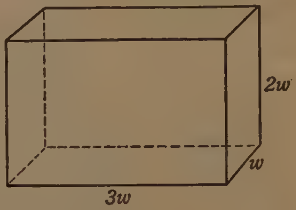
(1) If $w = 5$ in., find p.

(2) If $w = 2$ ft., find p.



7. The figure represents a model of a rectangular solid. Its dimensions are w, $2\,w$, and $3\,w$. If the total length of the edges is called t, evaluate t for the given values of w.

$w =$	2	4	$5\frac{1}{2}$	10	1
$t =$					



8. Many articles used in everyday life are put up by packing companies in packages or cans of uniform size and weight. The net weight is usually shown on the outside label. Suppose that you buy groceries from a wholesale house in dozen lots. If the net weight of one package is w, fill in the missing values in the following table:

If $w =$	3 oz.	5 oz.	8 oz.	1 lb.	1 lb. 4 oz.
$t = 12\,w =$					

10. Simplifying Formulas. Dissimilar Terms. In the formulas given so far we combined only similar terms. We shall now consider also the case of formulas having *dissimilar terms.*

Suppose that Mr. Jones, on an automobile trip, covers m miles the first day, d miles the second day, $2\,m$ miles the third day, and $3\,d$ miles the fourth day. How far does he travel during the four days?

If t is the total mileage, we have $t = m + d + 2\,m + 3\,d$. This answer is not in the simplest form. We may shorten it by writing

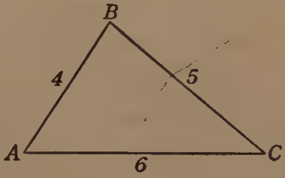
$$t = m + 2\,m + d + 3\,d,$$

or $$t = 3\,m + 4\,d.$$

Observe that the last form has but *two* terms, while the first form has *four* terms.

In this process of *simplifying* the result we have really used two important rules, which will now be explained.

11. Changing the Order of Terms. Suppose that the distances separating three towns, A, B, and C, are 4 miles, 5 miles, and 6 miles, as shown in the diagram. If a delivery wagon makes this circuit every day, how many miles does it cover each time?



If the wagon starts at A, its total mileage may be found by writing $t = 4 + 5 + 6$ or $t = 6 + 5 + 4$. If it starts at B, we have $t = 4 + 6 + 5$ or $t = 5 + 6 + 4$. And if it starts at C, we have $t = 6 + 4 + 5$ or $t = 5 + 4 + 6$. In other words, the total mileage may be found in *six* ways, the result being the same in each case.

From this illustration we infer the following

Rule of Order for Addition

In addition the addends may be combined in any order.

Thus, if a and b are to be added, we have

$$a + b = b + a.$$

12. Grouping of Terms. To find the sum of 4, 6, and 8, we may first combine 4 and 6 and then add 8, or we may add 4 to the sum of 6 and 8. In either case we form a *group* made up of two of the addends. That is,

$$4 + 6 + 8 = 4 + (6 + 8) = (4 + 6) + 8.$$

The symbols inclosing each of the groups are called *parentheses*.

From the above illustration we infer the following

Rule of Grouping for Addition

In addition the addends may be grouped in any manner.

Thus, $a + b + c = (a + b) + c = a + (b + c).$

NOTE. The Rule of Order will sometimes be referred to by the capital letter O, and the Rule of Grouping by G. Parentheses were introduced into algebra by a Fleming, Albert Girard, in 1629. They may have a variety of forms. Thus, instead of the ordinary parentheses, one may use brackets [ ] or braces { } for the same purpose.

The importance and use of these two rules will become clear from the following simple examples:

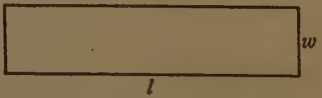
Example 1. Find the sum of 49, 6, 1, and 4.

The sum is $49 + 6 + 1 + 4$. Hence we have the

$$\begin{aligned} \text{Sum} &= 49 + 1 + 6 + 4 && \text{[By the Rule of Order]} \\ &= (49 + 1) + (6 + 4) && \text{[By the Rule of Grouping]} \\ &= 50 + 10 \\ &= 60. \end{aligned}$$

All rapid addition in arithmetic is based on these two rules.

Example 2. A rectangular field is to be inclosed by a fence. If the dimensions, in feet, are l and w, how many feet of fence are necessary?



Evidently, $p = l + w + l + w$. Hence we have

$$p = l + l + w + w, \qquad \text{[By the Rule of Order]}$$

and therefore $p = (l + l) + (w + w)$. [By the Rule of Grouping]

Hence $p = 2\,l + 2\,w$. [By the Rule of Similar Terms]

This work may be condensed as follows:

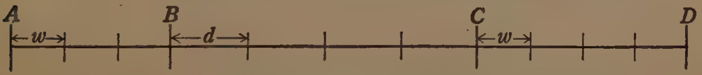
$$\begin{aligned} p &= l + l + w + w. && \text{[By O.]} \\ p &= (l + l) + (w + w). && \text{[By G.]} \\ p &= 2\,l + 2\,w. && \text{[By S. T.]} \end{aligned}$$

Exercises. *Dissimilar Terms*

Numbers 1 to 8, oral

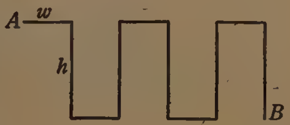
In the following exercises apply the rules given above, whenever it is possible:

1. Simplify the formula $t = a + b + 2a$. 3A + B
2. Simplify the formula $t = 2d + 5r + 7d + 9r$. = 90 + 14
3. Simplify the formula $s = 9n + 7c + 10n + 20c$.
4. Combine $4a + 5b + 6a + 7b$. (4A + 6A) + (5B + 7B
5. Combine $3w + 4d + 5w + 6d + 9w + 10d$.
6. Combine $100a + 97b + 6a + 3b + 10a$.
7. Combine $6.2e + 8.9d + 7.8c + 11.1d$.
8. Find the length of AD in the following diagram.



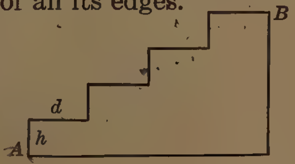
9. Three automobiles travel at hourly rates of d, l, and w respectively, all distances being expressed in miles. Find the total distance they cover in five hours.

10. The dimensions of a rectangle are $2w$ and $3h$. Find its perimeter p.

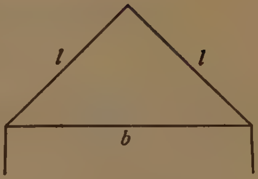


11. The accompanying design is called a *fret*. Observe how often w and h are repeated in the design. Find its total length from A to B.

12. The dimensions of a rectangular solid are l, w, and h. Find a formula for the total length of all its edges.

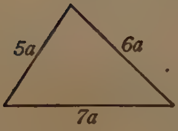


13. The figure shows the end view of a staircase. How long is a carpet covering this staircase from A to B?

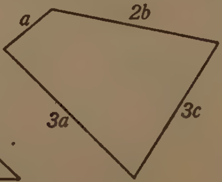


14. Find a formula for the perimeter of the gable triangle shown in the figure.

15. Write a formula for the perimeter of each of the following figures:



(1)



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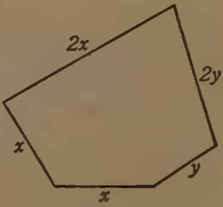


(3)

Evaluation

Explanation. The process of evaluation explained on page 19 may be extended to the case of formulas containing several letters. Thus, if in the formula $p = 2\,l + 2\,w$ we know that $l = 5$ and that $w = 3$, we can find the value of p as follows:

$$p = 2 \times 5 + 2 \times 3,$$

or $$p = 10 + 6;$$

that is, $$p = 16.$$

Whenever it is required to evaluate a formula, proceed as follows:

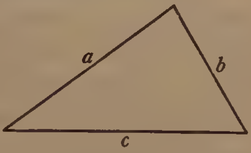
1. *Substitute for the letters their given numerical values.*

2. *Simplify the result.* In doing so, carry out the indicated multiplications or divisions first, and then add or subtract as required.

16. Using the formula $p = 2\,l + 2\,w$, fill in the missing values in the following table:

If $l =$	7	10	30	4.2
and $w =$	4	6	20	2.1
then $p =$	22			

17. Using the formula $t = 4\,l + 4\,w + 4\,h$, find t if $l = 5$, $w = 4$, and $h = 3$.



18. If the sides of a triangle have the lengths a, b, and c respectively, interpret the formula $p = a + b + c$.

19. Using the formula $p = a + b + c$, find p when $a = 20$, $b = 24$, and $c = 30$.

20. Using the formula $t = 10\,a + 5\,b$, find t if $a = 9$ and $b = 4$.

21. Find the value of each of the following expressions, given that $a = 1,\ b = 2,\ c = 3,\ d = 4$:

(1) $4\,a + 6\,b + 7\,c + 8\,d$.
(2) $90\,a + 70\,b + 80\,c + 40\,d$.
(3) $\frac{1}{2}\,a + \frac{1}{4}\,b + \frac{1}{6}\,c + \frac{1}{8}\,d$.
(4) $8.4\,a + 9.7\,b + 5.9\,c + 7.6\,d$.

22. The widths of two rectangles are $2\,w$ and $4\,w$, while their heights are $3\,h$ and $7\,h$ respectively. If $w = 3$ and $h = 2$, how much larger than the perimeter of the first rectangle is that of the second?

23. If $a = 2$, $b = 3$, and $c = 4$, find the value of $8\,a + 7\,b - 3\,c$.

24. If $r = 10$, $s = 7$, and $t = 4\frac{1}{2}$, find the value of $5\,r - 7\,s + 8\,t$.

25. Combine $10\,a + 6\,a + a - 9\,a + 15\,a + 7\,a - 20\,a$.

26. Combine $3\,x + 4\,y + 7\,x + 5\,y - 6\,x + 9\,y - 7\,y$.

27. The number of pupils enrolled in the academic course of a certain school is a; in the commercial course, c; in the technical course, t. Suppose that the annual cost per pupil to the school system is \$100, \$140, \$160, respectively, in these three courses. How many dollars must the school budget allow for this particular school?

13. The Fundamental Operations. You will recall that the fundamental operations of arithmetic are addition, subtraction, multiplication, and division. These operations are often denoted by the abbreviations A., S., M., D., respectively. Since they are also used constantly in algebra, a review of certain important terms is suggested at this point.

The following table shows that in the case of each of these processes the work may be arranged either *vertically* or *horizontally*.

ADDITION

(1)
$$\begin{array}{rl} 8\,a & \text{Addend} \\ 2\,a & \text{Addend} \\ \hline 10\,a & \text{Sum} \end{array}$$

(2) $8\,a + 2\,a = 10\,a$

SUBTRACTION

(1)
$$\begin{array}{rl} 8\,a & \text{Minuend} \\ 2\,a & \text{Subtrahend} \\ \hline 6\,a & \text{Difference} \end{array}$$

(2) $8\,a - 2\,a = 6\,a$

MULTIPLICATION

(1)
$$\begin{array}{rl} 2\,a & \text{Multiplicand} \\ 7 & \text{Multiplier} \\ \hline 14\,a & \text{Product} \end{array}$$

(2) $7 \times 2\,a = 14\,a$

DIVISION

(1)
$$\frac{10\,a \quad \text{Dividend}}{5 \quad \text{Divisor}} = 2\,a \quad \text{Quotient}$$

(2) $10\,a \div 5 = 2\,a$

The expression $7 \times 2\,a$ is often written in the form $7 \cdot 2\,a$. Observe that the dot is placed somewhat higher than a decimal point in order to avoid confusion. The expression $\frac{10\,a}{5}$ may also be written $\frac{1}{5} \times 10\,a$.

The sum of a literal expression and an arithmetic number is indicated by writing a *plus* sign between them. Thus, if n is *increased* by 5, we have $n + 5$. Similarly, if n is *diminished* by 5, we have $n - 5$.

In the following exercises the fundamental operations appear in simple combinations which illustrate still further the use of letters and numbers in algebra.

Practice Exercises. *Fundamental Operations*

Try to complete all the exercises in 10 minutes. Write answers only.

1. Add: $\begin{array}{r} 7\,h \\ \underline{5\,h} \end{array}$ $\quad \begin{array}{r} 20\,d \\ \underline{70\,d} \end{array}$ $\quad \begin{array}{r} 90\,h \\ \underline{h} \end{array}$ $\quad \begin{array}{r} 17.1\,x \\ \underline{5\quad x} \end{array}$ $\quad \begin{array}{l} 4\tfrac{1}{2}\,y \\ \underline{7\quad y} \end{array}$

2. Subtract: $\begin{array}{r} 20\,a \\ \underline{14\,a} \end{array}$ $\quad \begin{array}{r} 17\,n \\ \underline{14\,n} \end{array}$ $\quad \begin{array}{r} 8\,x \\ \underline{x} \end{array}$ $\quad \begin{array}{r} 19.7\,x \\ \underline{14.9\,x} \end{array}$ $\quad \begin{array}{l} 7\tfrac{1}{3}\,y \\ \underline{2\quad y} \end{array}$

Write the products which are indicated below:

3. $5 \times 6\,a$.
4. $9 \times 7\,b$.
5. $10 \times 14\,h$.
6. $8 \times 15\,d$.
7. $10 \cdot 4\,x$.
8. $12 \cdot 5\,a$.
9. $20 \cdot 3\,a$.
10. $18 \cdot 2\,d$.
11. $12 \cdot \frac{2}{3}\,h$.
12. $16 \cdot \frac{3}{4}\,n$.
13. $20 \cdot \frac{4}{5}\,x$.
14. $60 \cdot \frac{5}{6}\,w$.
15. $10 \times 5.1\,a$.
16. $100 \times 4.2\,x$.
17. $50 \times 4.5\,h$.
18. $10 \times 9.2\,h$.

Write the quotients which are indicated below:

19. $12\,a \div 3$.
20. $16\,w \div 4$.
21. $20\,x \div 5$.
22. $36\,n \div 12$.
23. $\frac{24\,h}{6}$.
24. $\frac{4.8\,n}{12}$.
25. $\frac{5.04\,x}{2}$.
26. $\frac{100\,n}{25}$.
27. $\frac{360\,n}{72}$.
28. $\frac{65\,x}{13}$.

29. Find $\frac{5}{6}$ of $60\,w$.
30. Divide $4.82\,h$ by .2.
31. $\frac{3}{4} \cdot 28\,y = ?$
32. $3\frac{1}{7} \cdot 21\,a = ?$
33. Combine $n + n + 1 + n + 2 + n + 3$.
34. Combine $2\,n + 2\,n + 2 + 2\,n + 4 + 2\,n + 6$.
35. Simplify $t = 8\,d + 7 - 4\,d + 6 + 2\,d - 5 + d$.
36. Simplify $t = n + 3 + 2\,n + 6 + 9\,n + 12$.
37. Simplify $t = w + 7 + 6\,w - 5 - 2\,w + 10$.

Oral Exercises. *Applying the Fundamental Operations*

1. If the present age of a boy, expressed in years, is a, what will be his age one year from now?

Solution. Since the value of a must be increased by 1, his age a year from now will be $a + 1$.

2. What was the boy's age a year ago?

Solution. A year ago he was 1 year younger. Hence his age at that time was $a - 1$.

3. The temperature this morning was t degrees. What is the temperature now if in the meantime it has risen 5 degrees?

4. If the present age of a boy, in years, is a, what was his age 3 years ago?

5. The speed of one train is s miles per hour. Another train travels 4 miles faster per hour. Find its speed.

6. The weight of a boy, in pounds, is w. Find the weight after he has gained 5 pounds.

7. The height of a room, in feet, is h. Find the height of another room which is 2 feet higher.

8. The length of a box, in inches, is l. Find the length of another box which is 10 inches shorter.

9. The weight of a baby, in pounds, is w. If it gains 4 ounces in weight, what is the resulting weight?

SUGGESTION. A weight of w pounds, expressed in ounces, becomes $16w$.

10. If a boy's age, in years, is a, what will it be 10 years from now? What was it 5 years ago? What is the age of a boy five times as old? 5 years more than twice as old? half as old?

11. A man's capital, in dollars, is denoted by c. How much has he if he gains \$1000 more? if he loses \$50? How much has he after losing one third of his capital?

EXPLANATION. In combining letters and numbers we may have expressions such as the following:

1. $n + 2$, namely, the *sum* of n and 2.
2. $n - 4$, the *difference* between n and 4.
3. $7n$, the *product* of 7 and n.
4. $\frac{n}{5}$, or $n \div 5$, or $\frac{1}{5}n$, the *quotient* of n divided by 5.
5. $3(n + 4)$, three times the sum of n and 4.
6. $\frac{1}{2}(n + 7)$, one half the sum of n and 7.
7. $\frac{1}{4}(n + 5)$, or $\frac{n+5}{4}$, or $(n + 5) \div 4$, the sum of n and 5, divided by 4.
8. $4n - 3$, the difference between $4n$ and 3.
9. $4(n - 3)$, four times the difference between n and 3.

12. In the following tables supply in the second column the values corresponding to those given in the first column:

n	$n+2$
3	
4	
9	
$1\frac{1}{2}$	
2.7	

n	$n-4$
7	
8	
20	
9.4	
$7\frac{1}{2}$	

n	$4n$
0	
1	
2	
7	
$2\frac{1}{2}$	

n	$\frac{n}{3}$
6	
9	
8	
9.3	
$\frac{1}{2}$	

13. A boy's score in a game is denoted by s. What is his score if he wins 8 more points? if he loses 2 points? The scores of five boys are s, $3s$, $4s$, $6s$, and $7s$ respectively. What is their total score?

14. If n denotes any number, give an expression for a number twice as large; half as large; 5 larger; 7 smaller.

15. If the length of a line is $l + 4$, what is the total length of two such lines? of three such lines?

14. Summary. In this chapter you have learned

(1) The use of letters in algebra.
(2) The meaning of certain new words.
(3) The Rule of Similar Terms.
(4) The Rule of Order for Addition.
(5) The Rule of Grouping for Addition.
(6) How to write certain formulas, such as $p = 4w$.
(7) How to evaluate formulas and algebraic expressions.
(8) How to combine letters and numbers by the four fundamental operations.

Test on Chapter I

1. Complete each of the following statements:

(1) In the term $7h$ the number 7 is called the ______.
(2) Terms such as $10d$ and $4d$ are called ______.
(3) To unite the terms of the expression $5l + 4l - 3l$ we make use of the rule of ______.
(4) The process of replacing a letter in an algebraic expression by a number is called ______.
(5) In combining the terms $2w + 3h + 5w + h$ the rules of ______, ______, and ______ are used.

2. Combine the similar terms in the following:

(1) $d + d + d + d$.
(2) $8w + 6w$.
(3) $3a + 6a + a$.
(4) $10r - 6r$.
(5) $12\frac{1}{2}p + 3\frac{3}{4}p$.
(6) $7w + 9w - 3w + 6w$.
(7) $3h + 6w + 4h + 5w$.
(8) $8a + 3n - 2a + n$.
(9) $3.2d + 14.5d + .3d$.
(10) $6\frac{1}{2}c + 3\frac{1}{4}d + \frac{3}{4}d + 3\frac{1}{2}c$.

3. Find the sum of the terms in each case:

(1)	(2)	(3)	(4)	(5)
$12w$	$5a$	$15n$	$10\frac{1}{4}h$	$1.5d$
$4w$	$8a$	$6n$	$4\ h$	$.7d$
w	$3a$	n	$3\frac{3}{4}h$	$3.3d$

4. Find the difference of the terms in each case:

(1) $\begin{array}{r} 8h \\ \underline{3h} \end{array}$ (2) $\begin{array}{r} 6c \\ \underline{c} \end{array}$ (3) $\begin{array}{r} 10\ n \\ \underline{3\frac{1}{2}\ n} \end{array}$ (4) $\begin{array}{r} 6\frac{3}{4}\ w \\ \underline{2\frac{1}{2}\ w} \end{array}$ (5) $\begin{array}{r} 6.9\ r \\ \underline{1.5\ r} \end{array}$

5. Find the product in each of the following:

(1) $\begin{array}{r} 4d \\ \underline{2} \end{array}$ (2) $\begin{array}{r} 3n \\ \underline{8} \end{array}$ (3) $\begin{array}{r} 6a \\ \underline{2\frac{1}{2}} \end{array}$ (4) $9w \times 3$. (5) $7 \times 4h$.

6. Find the quotient in each of the following:

(1) $\frac{20b}{4}$. (2) $\frac{9c}{9}$. (3) $\frac{2.4n}{8}$. (4) $60w \div 12$. (5) $3.68p \div 4$.

7. Given the formula $t = 8w$, make a table like the following and fill in the missing numbers:

If $w =$	2	3	10	$\frac{1}{2}$	$\frac{3}{4}$
then $t = 8w =$					

8. Given the formula $t = w + 3$, make a table like the following and fill in the missing numbers:

If $w =$	5	7	12	$\frac{1}{4}$	1.5
then $w + 3 =$					

$1\frac{3}{4}\ p$
$1\frac{1}{2}\ p$
$4\frac{1}{2}\ p$
p
$8\frac{3}{4}$

9. The sides of two squares are s and $3s$ respectively. Write a formula for the sum t of their perimeters.

10. A man's profits for four consecutive years are represented by p, $1\frac{3}{4}\ p$, $1\frac{1}{2}\ p$, and $4\frac{1}{2}\ p$ respectively.

(1) Find the total profit for four years.

(2) How much more did he make during the fourth year than during the third?

CHAPTER II

THE USE OF THE FORMULA

15. The Magic of the Formula. When a modern business block is to be erected, every detail of the work must be planned in advance, before the actual construction can begin. The exact dimensions of every part of the building must be indicated on the plans. Then a considerable amount of computing, or "figuring," must be done. Thus, it is necessary to find out how much excavating is required, and exactly how much steel is needed for the framework. The size and the strength of each steel beam must be determined. The same is true of all the other materials which are required to complete the building.

After the exact cost of the building has been determined, it becomes necessary to compute the interest charge which such an investment represents, as well as the operating expenses caused by such a building.

In short, at every turn an enterprise of this kind involves a good deal of mathematical work. Much of this work is simplified by the use of *formulas*. With the aid of *mathematical formulas*, engineers, contractors, and business men solve quickly and accurately, by a sort of magic, such problems as were mentioned above.

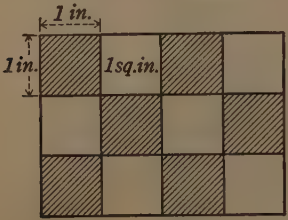
In the following pages we shall study a few of the simplest formulas used in mensuration and in business. Try to find out in each case how the particular formula that you are studying may be applied in actual life situations.

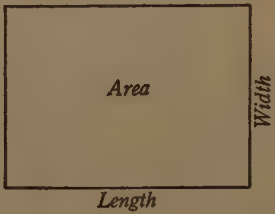
The new skyline of Chicago. The construction of skyscrapers such as those shown in the picture requires the extensive use of mathematical formulas

1. FORMULAS OF MENSURATION

16. Area of a Rectangle. If a rectangle is 4 inches long and 3 inches wide, its area is 4 × 3, or 12 square inches. Similarly, if the dimensions of a rectangle are expressed in terms of the same unit (for example, inches, feet, yards, centimeters), *the area equals the product of the numbers which measure the length and the width.* In brief form, this rule may be written

Area = length times width.





Using algebraic shorthand, we obtain the following formula for the area of *any* rectangle:

$$A = l \times w,$$

or

$$A = lw.$$

Observe that when two letters are written side by side in algebra, it means that we are to *multiply* together the numbers which they represent.

Illustrative Example. Find the area of a floor which is 20 feet long and 12 feet wide.

Solution. Using the formula $A = lw$, we replace l by 20 and w by 12. Hence we have $A = 20 \times 12 = 240$; that is, the area is 240 square feet. We have thus *evaluated* the formula $A = lw$.

NOTE. If the dimensions of a rectangle are 4 and 3, its area may obviously be found by writing either 4 × 3 or 3 × 4, the result being 12 in either case. That is, *the factors in a product may be written in any order*. Hence, the formula $A = lw$ may also be written $A = wl$.

Exercises. *Rectangles*

1. Using the formula $A = lw$, find A if $l = 24$ and $w = 14$.

Solution. Substituting 24 for l and 14 for w, we have

$$A = 24 \times 14, \quad \text{or} \quad A = 336.$$

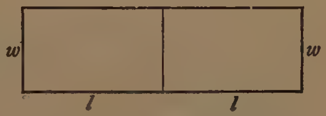
2. Using the formula $A = lw$, find A if

(1) $l = 20$ ft., $w = 10$ ft.
(2) $l = 3$ yd., $w = 2$ yd.
(3) $l = 20$ in., $w = 6$ in.
(4) $l = 14$ cm., $w = 5$ cm.

3. Using the formula $A = lw$, find the value of w if

(1) $A = 20$, $l = 5$.
(2) $A = 100$, $l = 25$.
(3) $l = 6$, $A = 12$.
(4) $l = 8$, $A = 15$.

4. Place two equal rectangular sheets of paper side by side, as shown in the figure. What is the total area of the new rectangle thus formed?



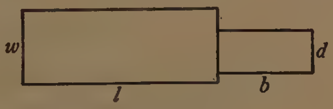
Solution. Evidently, $T = lw + lw$.

This may be written $T = 2\,lw$.

5. What is the combined area of the floor and the ceiling of a rectangular room, if their dimensions are b and h?

6. In the annexed figure, what is the formula for the total area?

$T =$ ----- + ------.

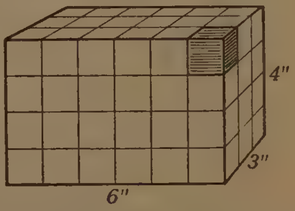


7. Explain in words, or by a figure, the meaning of the following expressions:

(1) $4\,lw$.
(2) $\dfrac{lw}{5}$.
(3) $4\,lw + 3\,lw$.
(4) $6\,lw - 4\,lw$.
(5) $3 \times 4\,lw$.
(6) $\dfrac{6\,lw}{3}$.

8. In the formula $A = lw$, how does the value of A change (1) if l is doubled? (2) if w is doubled? (3) if both l and w are doubled?

17. Volume of a Rectangular Solid. The figure represents a rectangular solid 6 inches long, 3 inches wide, and 4 inches high. We may imagine this box made up of layers of equal *cubes*, as shown in the figure. Evidently it contains $6 \times 3 \times 4$ of these cubes. Hence the volume is 72 cubic inches. Similarly, *the volume of any rectangular solid is the product of the length, the width, and the height.* This rule may be expressed in short form as follows:



$$V = l \times w \times h,$$

or

$$V = lwh.$$

Illustrative Example. Find the volume of a box which is 12 inches long, 6 inches wide, and 4 inches high.

Solution. Using the formula $V = lwh$, we have $l = 12$, $w = 6$, and $h = 4$, and thus obtain $V = 12 \times 6 \times 4 = 288$; that is, the box has a volume of 288 cubic inches.

NOTE. Since the factors of a product may be multiplied in any order, the formula $V = lwh$ may also be written in other forms, such as $V = wlh$, or $V = hlw$.

Exercises. *Rectangular Solids*

Using the formula $V = lwh$, find the missing values in each of the following exercises:

1. $l = 5$ in., $w = 4$ in., $h = 3$ in.

Solution. Substituting 5 for l, 4 for w, and 3 for h, we have

$$V = 5 \times 4 \times 3, \quad \text{or} \quad V = 60 \text{ cubic inches.}$$

2. $l = 24$ ft., $w = 20$ ft., $h = 12$ ft.
3. $V = 70$ cu. ft., $l = 35$ ft., $w = 2$ ft.
4. $V = 160$ cu. ft., $w = 8$ ft., $l = 10$ ft.

5. $w = 5$ ft., $V = 90$ cu. ft., $h = 3$ ft.

6. $h = 2$ ft., $l = 6$ ft., $V = 60$ cu. ft.

7. If you know the volume of a rectangular solid, and two of the dimensions, how do you find the third dimension?

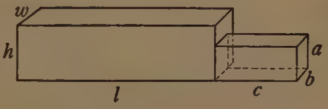
8. A storeroom is to be 12 feet long and 10 feet wide. If it is to contain 1000 cubic feet, how high must it be?

9. Using the formula $V = lwh$, find the value of h if

(1) $V = 320$ cu. ft., $l = 10$ ft., $w = 8$ ft.
(2) $V = 180$ cu. ft., $w = 5$ ft., $l = 9$ ft.
(3) $l = 20$ in., $V = 1400$ cu. in., $w = 10$ in.
(4) $l = 12$ in., $V = 1200$ cu. in., $w = 10$ in.

10. If the dimensions of a rectangular brick are l, w, and h, what is the combined volume of 50 bricks?

11. In the accompanying figure, what is the formula for the total volume?



12. Explain in words, or by means of a diagram, the meaning of the following expressions:

(1) $3\,lwh$. (3) $6\,abc + 3\,abc$. (5) $3 \times 2\,lwh$.
(2) $\frac{lwh}{5}$. (4) $7\,bdh - 2\,bdh$. (6) $lwh \div 4$.

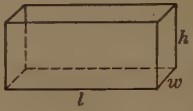
13. A cellar is to be excavated to a depth of 6 feet. If the ground plan of the house is 28 feet by 40 feet, how many cubic yards of earth must be removed?

14. In the formula $V = lwh$, how does the value of V change (1) if l is doubled? (2) if w is doubled? (3) if h is doubled?

15. If each of the dimensions of a rectangular solid is doubled, what is the effect on the volume?

Exercises. *Rectangles and Rectangular Solids*

1. If the dimensions of a rectangular box are l, w, and h, write a formula for



(1) the bottom of the box;
(2) the front of the box;
(3) the total surface of the box.

2. Mr. Jones wished to build a coal bin that would hold 10 tons of coal when filled to a height of not more than 6 feet. If one ton of coal occupies about 35 cubic feet of space, how large a floor space or area must the coal bin have in order to accommodate just 10 tons?

THINK: $6 \times (\qquad) = 10 \times 35.$

3. Carl Elliott's father owns six building lots, each 50 ft. × 150 ft. Carl asked how the total area of these lots compared with one acre. He learned that one acre contains 43,560 square feet. Can you answer his question?

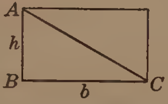
4. A business block covered 27,000 square feet of land. Its foundation reached to a depth of 40 feet in the ground. How many cubic yards of earth had to be removed? What did it cost to remove this material at 90 cents a load? (Allow 1 cubic yard for each load.)

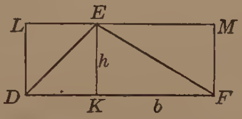
5. How many cubic feet of air space are there in each of the following schoolrooms?

(1) Classroom	22 ft. × 29 ft. × 12 ft.
(2) Library	28 ft. × 77 ft. × 12 ft.
(3) Gymnasium	60 ft. × 90 ft. × 25 ft.

6. A brick wall, 200 feet long, 2 feet in thickness, and 8 feet high, rests on a concrete foundation. If ordinary brickwork weighs 112 pounds per cubic foot, what is the weight supported by the concrete foundation?

18. Area of a Triangle. A study of the two adjoining figures will serve to make clear the following rule: *The area of a triangle is equal to half the product of its base and its height.* This rule may be shortened by means of the formula





$$A = \frac{1}{2}\,bh, \quad \text{or} \quad A = \frac{bh}{2}.$$

all one formula

Exercises. *Triangles and Irregular Figures*

Using the formula $A = \frac{1}{2}\,bh$, *find the missing number in each of the following exercises:*

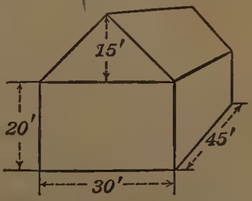
1. $b = 4$, $h = 6$.
2. $b = 9$, $h = 12$.
3. $b = 16$, $h = 17\frac{1}{2}$.
4. $A = 20$, $b = 10$.
5. $A = 100$, $h = 20$.
6. $b = 20$, $h = 10$.
7. $b = 8$, $h = 3$.
8. $b = 7.2$, $h = 5$.
9. $b = 16\frac{1}{2}$, $h = 8$.
10. $A = 25$, $b = 8$.

11. What is the area of a triangle if its base is $4\,w$ and its height is 20?

12. What is the area of a triangle if its height is $6\,h$ and its base is 8?

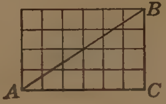
13. What is the height of a triangle if its area is $14\,n$ and its base is 4?

14. A house has a width of 30 feet and a length of 45 feet. The height to the eaves is 20 feet. The gable triangle is 15 feet high. Find the area of each gable triangle.

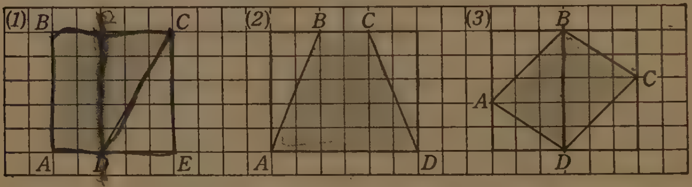


15. In Exercise 14, find the entire outer area of the house, except the area of the roof.

16. In the figure, find the area of the triangle *ABC*.

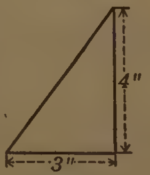


17. By using squared paper, and applying the formulas for the area of a rectangle and of a triangle, it is often possible to find the areas of *irregular figures* such as the ones shown in the following diagrams:

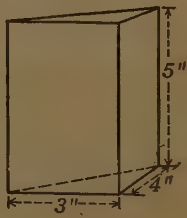


Thus, in the first diagram, find the area of rectangle *ABCE*, and then subtract the area of triangle *CDE*. The result is the area of the irregular figure *ABCD*. Find the area of *ABCD* in each case.

19. Volume of a Prism. The figure at the right below represents a triangular prism. The base is a right triangle whose area is $\frac{1}{2} \times 3 \times 4$, or 6 square inches.

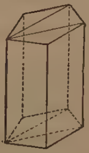


Base



If a vessel having the shape and the interior dimensions shown in the figure were filled with water to a height of 1 inch, it is evident that the volume of the water would be 6 cubic inches. Hence, since the prism is 5 inches high, it is obvious that the volume of the entire prism is five times as large; that is, 5×6, or 30 cubic inches.

This explanation could easily be extended to the case of any prism. That is,



The volume of any prism may be found by multiplying the area of the base by the height.

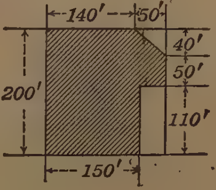
This rule may be expressed by the formula

$$V = Bh.$$

Exercises. *Volumes of Prisms*

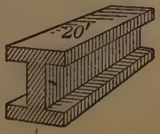
1. The base of a prism is a triangle whose area is 24. The altitude of the prism is 12. Find its volume.

2. The base of a prism has an area of 21 square feet. If the altitude of the prism is 10 feet, what is its volume?



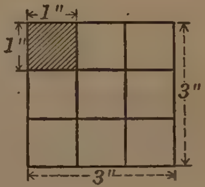
3. The diagram shows the ground plan of a business block. If the excavation for its foundation is to be 18 feet in depth, how many cubic yards of earth must be removed?

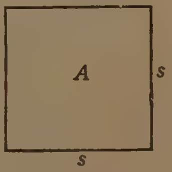
4. A steel beam of the form shown has a cross section containing 14.1 square inches. Its length is 20 feet. What is the weight of such a beam if steel weighs about 490 pounds per cubic foot? Show that the weight is approximately half a ton.



5. The base of a granite column like the one shown in the figure at the right was a hexagon having an area of 4.8 square feet. The height of the column was 10 feet. If granite weighs 170 pounds per cubic foot, what is the total weight of six such columns?

170





20. Area of a Square. The dimensions of a square are equal. If the side of a square is 3 inches, its area is 3 × 3, or 9 square inches. The product 3 × 3 is often written 3^2. This is read "the square of three," or "three square." Hence the formula for the area of a square of side s becomes

$$A = s \times s,$$

or

$$A = s^2.$$

NOTE. The small figure 2, in the expression s^2, is called an *exponent.*

Exercises. *Squares*

1. Review the table of square measure. How many square inches in a square foot? How many square feet in a square yard? How many square millimeters in a square centimeter? (1 centimeter = 10 millimeters.)

2. What is the value of 4^2? 5^2? 7^2? 9^2? 10^2?

3. What is the meaning of b^2? of d^2?

Using the formula $A = s^2$, find the value of A in each of the following exercises:

4. $s = 12.$
5. $s = 25.$
6. $s = 31.$
7. $s = 7.2.$
8. $s = 19.4.$
9. $s = 12.1.$

10. Using the formula $A = s^2$, find s if $A = 64$.

THINK: Which number, multiplied by itself, gives 64?

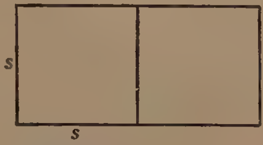
11. If $A = 144$; find s.

12. If $A = 400$; find s.

13. If two equal square pieces of paper are placed side by side, as shown in the figure, what is the area of the resulting rectangle?

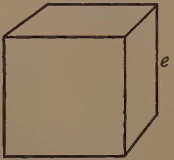
Solution. $T = s^2 + s^2$. This may be written

$$T = 2\,s^2.$$

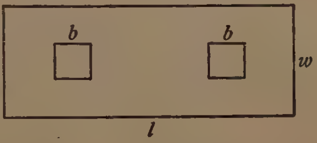


14. Draw a figure to illustrate the meaning of $3\,s^2$.

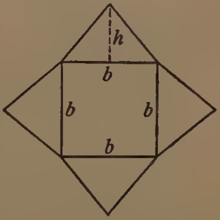
15. Each face of a cube is a square. If the edge of a cube is e, the area of one face is e^2. Write a formula for the total surface of the cube.



16. From a rectangular sheet of tin two equal square pieces are cut. Using the dimensions given in the accompanying figure, write a formula for the remaining area.



17. The figure represents a square and four adjoining triangles. If the height of each triangle is h, write a formula for the total area T of the figure. Using the values $b = 4$ and $h = 3$, find T.



In the following exercises, a, b, and c represent sides of squares. Explain in words, or by using a diagram, the meaning of each of the following expressions:

18. $4\,a^2$.

19. $6\,a^2 + 7\,a^2$.

20. $a^2 + b^2$.

21. $a^2 - b^2$.

22. $3 \times 4\,a^2$.

23. $8\,a^2 \div 2$.

24. $a^2 + b^2 + c^2$.

25. $a^2 + b^2 - c^2$.

26. $3\,a^2 - 2\,b^2$.

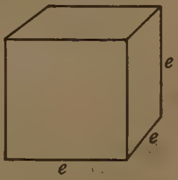
27. $4\,a^2 + 6\,b^2$.

28. In Exercises 18–27, if $a = 4$, $b = 3$, and $c = 2$, find the numerical value of each expression.

EXPLANATION. In evaluating expressions such as those given above, find *first* the squares of the numbers substituted for the letters, *then* perform the indicated multiplications or divisions, and *finally* add or subtract as required.

Thus,
$$\begin{aligned} 3a^2 - 2b^2 &= 3 \times 4^2 - 2 \times 3^2 = 3 \times 16 - 2 \times 9 \\ &= 48 - 18 \\ &= 30. \end{aligned}$$

21. Volume of a Cube. The dimensions of a cube are all equal. Hence the volume is found by multiplying together three equal numbers. Thus, if V is the volume of a cube (in cubic units), and e represents the length of one edge (in linear units), then $V = e \times e \times e$. This is abbreviated as follows:



$$V = e^3.$$

The expression e^3 is read "the cube of e," or "e cube."

NOTE. The small figure 3 in the expression e^3 is an *exponent*.

Exercises. *Cubes*

1. Review the table of cubic measure. How many cubic feet in a cubic yard? How many cubic inches in a cubic foot?

2. Find the volume of a cube if its edge is 4 inches.

Solution. Since $V = e^3$,

we have $V = 4^3$, or $4 \times 4 \times 4$;

that is, $V = 64$ cubic inches.

Find the volume of a cube which has an edge of

3. 5 in.　4. 2 ft.　5. 3 yd.　6. 3 cm.

7. How many cubic feet are there in 3 cubic yards?

8. Compare the volumes of two cubes whose edges are 6 feet and 3 feet respectively.

9. Compare the volumes of two cubes whose edges are 9 feet and 3 feet respectively.

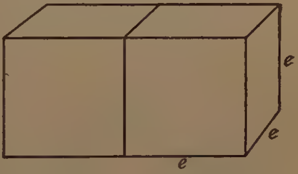
Using the formula $V = e^3$, find the value of V in each of the following exercises:

10. $e = 4$.
12. $e = 6$.
14. $e = 12$.
11. $e = 10$.
13. $e = 1$.
15. $e = 32$.

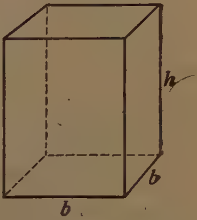
16. Using the formula $V = e^3$, find e if $V = 8$.

THINK: $e \times e \times e = 8$. Hence $e = 2$.

17. If $V = 125$, find e.



18. If two cubes, each of edge e, are placed side by side as shown in the figure, what is the volume of the resulting rectangular solid?



19. By means of models, or by a drawing, explain the meaning of $4e^3$.

20. The volume of the solid shown in the accompanying figure is $b \times b \times h$. This may be written b^2h. What is the meaning of $2b^2h$?

21. If a, b, and c are the edges of cubes, explain the meaning of the following expressions:

(1) $7a^3$.
(4) $3a^3 + 4b^3$.
(7) $4a^3 + 8b^3$.
(2) $a^3 - b^3$.
(5) $4a^3 \times 3$.
(8) $5a^3 - 2b^3$.
(3) $a^3 + b^3 + c^3$.
(6) $8a^3 \div 4$.
(9) $15b^3 \div 3$.

22. In Exercise 21, if $a = 3$, $b = 2$, and $c = 1$, find the numerical value of each expression. (The method to be followed here is similar to the one explained on page 46.)

Practice Exercises. *Fundamental Operations*

Try to complete all the exercises in 12 minutes. Write the answers only.

1. *Add:*

$$\begin{array}{rrrrr} 4\,x & 9\,ab & 30\,cde & 4.7\,a^2 & 8\,n^3 \\ 7\,x & 7\,ab & 41\,cde & 9.2\,a^2 & 7\,n^3 \\ \underline{5\,x} & \underline{10\,ab} & \underline{59\,cde} & \underline{10.5\,a^2} & \underline{4\,n^3} \end{array}$$

2. *Subtract:*

$$\begin{array}{rrrrr} 91\,w & 31\,a^2b & 47\,ab^2 & 50\,x^3 & 19\,y^2 \\ \underline{42\,w} & \underline{19\,a^2b} & \underline{29\,ab^2} & \underline{21\,x^3} & \underline{\,y^2} \end{array}$$

Multiply:

3. $10 \times 7 \times 3\,a$.
4. $2 \times 5 \times 2\,w$.
5. $8 \cdot 2\,cd$.
6. $12 \times 5\,a^2$.
7. $6 \times 5\,x^3$.
8. $8 \cdot 7\,a^2b$.

Divide:

9. $\frac{7\,w}{7}$.
10. $\frac{8\,a}{8}$.
11. $\frac{30\,n}{5}$.
12. $\frac{42\,x^2}{7}$.
13. $\frac{60\,x^3}{12}$.
14. $\frac{4.5\,n^2}{3}$.

15. Combine $9\,ab + 7\,ac - 4\,ab + 4\,ac + 10\,ab - ac$.

16. Combine $7\,x^2 + 9\,y^2 - 5\,x^2 + 3\,y^2 + 6\,x^2 - 8\,y^2$.

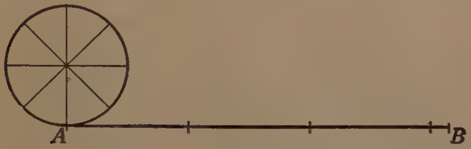
17. Combine $6\,a^3 + 14\,a^2b + 9\,ab^2 + 7\,a^2b - 6\,ab^2 - 3\,a^3$.

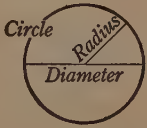
18. Using the formula $t = 3\,a + 5\,b + 6\,c$, find t if
(1) $a = 5$, $b = 7$, $c = 9$. (2) $a = 10$, $b = 12$, $c = 14$.

19. Using the formula $T = a^2 + b^2 + c^2$, find T if
(1) $a = 1$, $b = 2$, $c = 3$. (2) $a = 10$, $b = 9$, $c = 8$.

20. Using the formula $s = 7\,x^2 - 3\,y^2$, find s if
(1) $x = 3$, $y = 4$. (2) $x = 10$, $y = 5$.

22. Circumference of a Circle. To measure the circumference of any circular object, you may wind a piece of string or a sheet of paper around the object. It is then possible to mark off on the string or on the paper the desired circumference. You will find that, in each case, the circumference is about $3\frac{1}{7}$ times as long as the diameter of the circular object. That is, *the circumference is $3\frac{1}{7}$ times the diameter*. This rule may be shortened as follows:





$$C = 3\tfrac{1}{7} \times d.$$

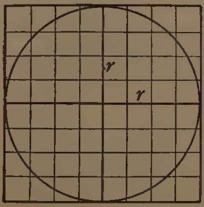
NOTE. The number $3\frac{1}{7}$ is only approximately correct. A more accurate value is 3.1416. Since no fraction or decimal exactly indicates how many times d is contained in C, it is customary to represent this result by the Greek letter π. Hence the above formula is often stated as follows:

$$C = \pi d.$$

If r represents the radius of the circle, the formula becomes

$$C = 2\,\pi r.$$

23. Area of a Circle. Suppose that a circle of radius r is drawn on squared paper, as shown in the diagram. The large square in which the circle is inclosed may then be divided into four squares, each having a side equal to r. The area of the large square is readily seen to be $4\,r^2$. The area of the circle is therefore less than $4\,r^2$. It can be shown that the area of the circle is $3\frac{1}{7} \times r^2$, approximately. The exact formula is



$$A = \pi r^2.$$

Exercises. *Circles*

1. Find the circumference of a circle with a diameter of 7 inches.

Solution. We have $C = \pi d = \frac{22}{7} \times 7 = 22$; that is, the circumference is 22 inches.

Find the circumference of a circle having a diameter of

2. 14 in.
3. 21 in.
4. 10 in.
5. 6.1 ft.
6. 23 ft.
7. 3.4 cm.

8. What is the area of a circle which has a radius of 3 inches?

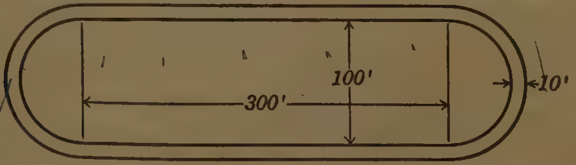
Solution. $A = \pi r^2 = \frac{22}{7} \times 3 \times 3 = \frac{198}{7} = 28.28 \cdots$; that is, to the nearest tenth of a square inch, the area is 28.3 square inches.

Find the area of a circle which has a radius of

9. 7 in.
10. 10 ft.
11. 8 yd.
12. 7.6 ft.
13. 8.4 in.
14. 11.2 cm.
15. 9 mm.
16. 100 ft.

17. A circular reservoir has a diameter of 400 feet. A fence is to be built around it. How many feet of fencing must be provided?

18. The circumference of a tree at a certain height was found to be 21 feet. Find the area of a cross section of the tree at that height.



19. A running track has the form and the dimensions shown in the figure above. Find the area of the track.

Illustrative Example. What is the interest on \$600, for 2 years, at 6%?

Solution. In this case $r = \frac{6}{100}$, $t = 2$, $p = 600$. Using the formula $i = prt$, we have $i = \overset{6}{\cancel{600}} \times \frac{6}{\cancel{100}} \times 2 = 72$; that is, the interest is \$72.

Exercises. *Formulas Used in Business*

1. What is the interest on \$700, for 3 years, at 5%?

2. What is the interest on \$400, for 2 years, at $5\frac{1}{2}$%?

SUGGESTION. Observe that $5\frac{1}{2}\% = \frac{11}{200}$.

3. Using the formula $i = prt$, find i in the following cases:

INTEREST	PRINCIPAL	RATE	TIME IN YEARS
(1) ---	\$720	6%	$3\frac{1}{2}$
(2) ---	\$950	5%	$2\frac{1}{3}$
(3) ---	\$1040	$6\frac{1}{2}$%	$1\frac{1}{4}$
(4) ---	\$2080	$4\frac{1}{2}$%	$\frac{3}{4}$
(5) ---	\$860	$5\frac{1}{2}$%	$\frac{1}{4}$

4. What is the interest on \$1060, for 3 months, at 6%?

SUGGESTION. Express 3 months as a fraction of a year.

5. It cost \$4,000,000 to put up a new bank building. If this money was borrowed at a rate of 5%, what was the interest charge against this building each year?

6. The *amount formula* is $A = p + prt$; that is, *the amount equals the principal plus the interest.* Find A, if $p = \$2500$, $r = 4\%$, and $t = 3$.

7. A man bought a \$10,000 home. He paid \$2000 in cash. For the balance he gave a mortgage, the interest rate being 6%. How much interest did he pay each year?

8. The *profit formula* is $p = s - c$; that is, *the profit equals the selling price minus the cost.* Find p, if $s = \$3.50$ and $c = \$1.40$.

9. The term "overhead expense" is often used to represent the extra expense caused by transporting, handling, and selling goods, from the time they are produced in the factory or on the farm to the time when they reach the consumer. If e denotes this extra expense, while s and c have the same meaning as in Exercise 8, we obtain the formula

$$s = c + e + p.$$

Express this formula in words. If $c + e$ is equal to s, what is the value of p?

If $c + p$ is equal to s, what is the value of e?

10. You have learned in arithmetic that *percentage equals base times rate.* Write a formula expressing this rule.

Using the formula $p = br$, find the missing values in the following exercises:

11. $b = 200$, $r = 7\%$.

12. $b = 1400$, $r = 15\%$.

13. $p = 20$, $b = 200$.

14. $p = 800$, $b = 2400$.

26. The Formula as an Expression of Relationships. Any formula of the kinds which have been considered thus far really represents *a shortened mathematical rule.* It may also be said that such a formula is a condensed statement of a mathematical *relation,* showing how certain quantities which may change in value *depend* on each other.

For example, the familiar formula $A = \pi r^2$, for the area A of a circle with radius r, expresses the relation between the radius of a circle and its area and shows exactly how the area of the circle *depends* on the length of the radius.

27. Summary. In this chapter you have learned

(1) Important formulas used in mensuration.
(2) Important formulas used in business.
(3) How to evaluate these formulas.
(4) How to express the meaning of these formulas in words.
(5) How to apply these formulas.

Supplementary Honor Work

The following miscellaneous group of formulas is taken principally from the fields of *industry* and *science*. It is not expected that their meaning will be explained or discussed, except at the discretion of the teacher.

Find the value of the letter indicated in each case by substituting the given values in the formula:

1. *Pulley* formula: $DS = ds$.
If $D = 5\frac{1}{2}$, $S = 400$, and $d = 26$, find s.

2. Formula from *electricity*: $A = \frac{V}{R}$.
If $V = 110$ and $R = 440$, find A.

3. *Safe load* formula for steel I-beam: $W = \frac{1695\,AD}{L}$.
If $A = 5.34$, $D = 8$, and $L = 16$, find W.

4. *Pressure* formula: $P = ahd$.
If $a = \frac{1}{144}$, $h = 72$, and $d = 62.4$, find P.

5. Formula for *speed of cutting tool*: $n = \frac{3.8\,s}{d}$.
If $s = 40$, and $d = 5$, find n.

6. An *airplane* formula: $R = KSV^2$.
If $K = .000541$, $S = 36$, and $V = 92$, find R.

7. *Power shaft* formula: $M = \frac{D^3R}{125}$.
If $D = 2$, and $R = 600$, find M.

8. Formula for *volume of sphere*: $V = \frac{1}{6}\pi d^3$.
If $d = 3$, find V.

9. Formula for *surface of a cylinder*: $S = 2\pi rh$. If $r = 7$ and $h = 12$, find S.

10. Formula from *electricity*: $W = \frac{V^2}{R}$.
If $V = 110$ and $R = 440$, find W.

11. *Energy* formula: $K = \frac{Wv^2}{2g}$.
If $W = 3$, $v = 40$, and $g = 32$, find K.

12. A formula for *resistance of a train*: $r = 3 + \frac{s}{6}$.
If $s = 40$, find r.

13. *Railroad grade-resistance* formula: $R = .379\,G$.
If $G = 50$, find R.

14. *Belting* formula: $W = T_1V - T_2V$.
If $V = 900$, $T_1 = 400$, and $T_2 = 160$, find W.

15. *Mechanical advantage* formula: $M = \frac{W}{E}$.
If $M = 2.5$ and $E = 15$, find W.

16. *Specific gravity* formula: $S = \frac{w_1}{w_1 - w_2}$.
If $w_1 = .365$ and $w_2 = .320$, find S.

17. A *chemical* formula: $n = \frac{s}{w} - 1$.
If $s = .18$ and $w = .03$, find n.

18. Formula for *heat loss*: $W = A^2R$.
If $A = 18$ and $R = 5$, find W.

19. Formula for *flow of air in pipe*: $Q = .327\,vd^2$.
If $v = 12$ and $d = 10$, find Q.

20. *Centrifugal force* formula: $F = \frac{Wv^2}{gr}$.
If $W = 1$, $v = 126$, $g = 32$, and $r = 3$, find F.

21. Formula for *distance* covered by a falling body: $s = \frac{1}{2}gt^2$.
If $g = 32$ and $t = 3$, find s.

Test on Chapter II

1. In each of the following cases several answers are given for the missing value. Select the one which you consider to be the correct result.

(1) $S = 4n + 6$. If $n = 3$, then $S =$ (18, 13, 36, 10).
(2) $V = e^3$. If $e = 2$, then $V =$ (6, 8, 9, 4).
(3) $A = \frac{bh}{2}$. If $b = 8$ and $h = 6$, then $A =$ (7, 16, 48, 24).
(4) $d = rt$. If $d = 40$ and $t = 5$, then $r =$ (8, 35, 200, 45).
(5) $i = prt$. If $p = 360$, $r = .05$, and $t = 2$, then $i =$ (18, 36, 7.2, 3600).

2. Find the value of the missing letter in each of the following:

(1) $V = lwh$. If $l = 2$, $w = 5$, and $h = 3$, then $V =$ ------.
(2) $A = s^2 - w^2$. If $s = 6$ and $w = 1$, then $A =$ ------.
(3) $v = \frac{d}{t}$. If $d = 60$ and $t = 5$, then $v =$ ------.
(4) $a = \frac{s}{n}$. If $a = 15$ and $n = 3$, then $s =$ ------.
(5) $s = \frac{1}{2}at^2$. If $a = 32$ and $t = 5$, then $s =$ ------.

3. Find the missing numbers in each of the following tables:

(1) $V = lwh$.

$l =$	8 ft.	19 yd.
$w =$	3 ft.	12 yd.
$h =$	$\frac{1}{2}$ ft.	$\frac{3}{4}$ yd.
$V = lwh =$		

(2) $S = 6e^2$.

$e =$	5 in.	$\frac{2}{3}$ yd.
$S = 6e^2 =$		

4. Write the formula which you would use to find

(1) the area of the floor of an ordinary room;
(2) the area of the top of a round table;
(3) the area of the gable triangle of a house;
(4) the length of the earth's equator;
(5) the volume of a paint can.

5. Express in words each of the following formulas:

(1) $A = \frac{bh}{2}$. (2) $i = prt$. (3) $V = \pi r^2 h$.

6. Copy the numbers (1) to (12) of Column A and write after each the number of the corresponding description in Column B.

COLUMN A	COLUMN B
(1) $A = lw$.	(1) Area of a square.
(2) $A = s^2$.	(2) Area of a triangle.
(3) $A = \frac{1}{2} bh$.	(3) Circumference of a circle.
(4) $p = br$.	(4) Area of a circle.
(5) $A = \pi r^2$.	(5) Area of a rectangle.
(6) $V = lwh$.	(6) Volume of a rectangular solid.
(7) $V = e^3$.	(7) Volume of a prism.
(8) $S = 6e^2$.	(8) Volume of a cylinder.
(9) $V = Bh$.	(9) Surface of a cube.
(10) $V = \pi r^2 h$.	(10) Simple-interest formula.
(11) $i = prt$.	(11) Volume of a cube.
(12) $C = 2\pi r$.	(12) Percentage formula.

CHAPTER III

THE MAKING OF FORMULAS

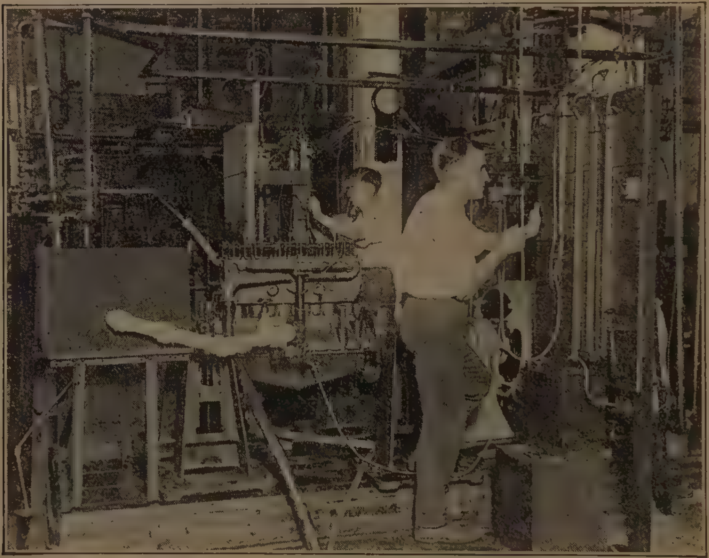
28. How Progress was made Possible by the Formula. If you could go back one hundred years and study the life of the people at that time, you would be amazed at the difference between conditions *then* and *now*. Above all, you would realize how many conveniences and improvements which we enjoy nowadays were entirely unknown only a few generations ago. Think of electric lights, telephones, radios, airplanes, typewriters, automobiles, steamships, movies, and the like. As everyone knows, these tremendous changes were made possible only by countless inventions and discoveries and by the use of modern machinery.

Of course, the processes underlying these discoveries required many years of patient experimentation in shops and laboratories. Thus, chemists worked for a long time before they discovered the "formula" for a rust-proof, stainless steel. Whenever the formulas based on such experiments are finally perfected, they are very valuable. They unlock doors which formerly were closed. As by magic, they create wealth and stimulate commerce and industry.

The formula is thus a storehouse of information. It is a key to knowledge. It saves time, money, and energy, for it often enables us to complete work quickly and easily which otherwise would be very tedious and expensive.

In this chapter we shall therefore try to learn more about the way in which formulas are constructed and discovered.

29. Obtaining Formulas from Verbal Statements. When a mathematical relation is given in verbal form, we may desire to express it algebraically. The following exercises will give practice in obtaining simple algebraic formulas which are based on verbal statements or descriptions.



A laboratory of the United States Bureau of Standards. The two men shown in the picture have been making thousands of tests in an effort to improve the starting of automobile and airplane engines in cold weather

Exercises. *From Words to Formulas*

Each of the following three groups of exercises is to help you in making and using an important formula.

THE DISTANCE FORMULA

1. If an automobile travels at the rate of 20 miles an hour, how far can it go in 5 hours?

2. A train travels at the rate of r miles an hour. How far does it travel in 5 hours? in t hours?

3. If r denotes the rate at which you walk, and t the time in hours, show that the distance which you cover is represented by the formula $d = rt$. Translate this formula into words.

4. Using the formula $d = rt$, find by substitution each of the missing values in the following exercises, if r represents the hourly rate in miles and t represents the time in hours:

(1) $r = 10$, $t = 2$.
(2) $r = 40$, $t = 6$.
(3) $d = 100$, $t = 5$.
(4) $d = 240$, $t = 8$.
(5) $r = 35$, $d = 70$.
(6) $r = 6$, $d = 50$.
(7) $r = 4\frac{1}{2}$, $t = 8$.
(8) $r = 16.1$, $t = 5$.

5. Show that the "distance formula" can be written in two other forms; namely, $r = \frac{d}{t}$, and $t = \frac{d}{r}$. Explain the meaning of each.

6. Sound travels about 1100 feet a second. How far will the report of an explosion travel in 6 seconds?

THE COST FORMULA

7. If eggs cost 60 cents a dozen, what is the cost of 7 dozen?

8. The cost of a tablet is c. What is the cost of 12 tablets? of n tablets?

9. If c denotes the cost of one pound of goods and n is the number of pounds purchased, show that the *total cost* may be represented by the formula $T = nc$. Express the formula in words.

10. Using the formula $T = nc$, find by substitution each of the missing values in the following exercises, if c represents the cost per pound and n represents the number of pounds:

(1) $c = 8$ cents, $n = 5$.
(2) $c = 40$ cents, $n = 10$.
(3) $T = \$2$, $n = 40$.
(4) $T = \$80$, $c = \$1$.
(5) $T = \$3.60$, $n = 9$.
(6) $T = \$20.80$, $c = 20$ cents.
(7) $c = \$1.20$, $n = 8\frac{1}{2}$.
(8) $c = \$4.50$, $n = 7$.

11. Show that the "cost formula" can be written in two other forms; namely, $n = \frac{T}{c}$ and $c = \frac{T}{n}$. Explain the meaning of each.

12. If it costs x dollars to educate a pupil for one year, how many children can be educated for \$2,000,000 a year?

The Formula for the Average

Explanation. John solved 8 examples on Monday and 12 examples on Tuesday. His average was $\frac{8 + 12}{2}$, or 10 examples, each day. In other words, to find the *average* of two numbers, you divide their sum by 2; to find the average of three numbers, you divide their sum by 3; and so on.

Evidently *the average of a set of numerical items is found by dividing their sum by the number of items considered.* Hence the formula for the average is

$$a = \frac{s}{n}.$$

13. Find the average of 3, 4, 5.

14. Find the average of 4, 5, 6, 9.

15. What is the average of 10, 12, 15, 30, 43?

16. If three numbers are denoted by b, c, and d, write a formula for their average.

17. Show that the formula $a = \frac{s}{n}$ may also be written in forms $s = a \times n$, and $n = \frac{s}{a}$. Explain the meaning of each.

18. The average daily lunch-room expense of the 257 pupils in a certain school was found to be 18 cents. What were the daily lunch-room receipts? Explain.

19. Suppose that John saved 6 cents a week for 4 weeks, 9 cents a week during the next 4 weeks, and 12 cents a week during the following 5 weeks. Find his average weekly savings during this period of time.

SUGGESTION. Find the total savings and divide by the total number of weeks.

20. If John saved 5 cents a week for b weeks, 7 cents a week for c weeks, and 9 cents a week for d weeks, write a formula for his average weekly savings.

Miscellaneous Exercises. *From Words to Formulas*

Numbers 1 to 20, oral

1. Express in inches: a feet; b yards; $2d$ feet.

2. Change into feet: x yards; a miles; d inches.

3. Express in inches: (1) a feet and 2 inches; (2) b yards and 5 inches.

Solution. (1) $(12a + 2)$ inches; (2) $(36b + 5)$ inches.

4. How many yards are there in c feet? in $2d$ miles?

5. Express in dollars: c cents; n quarters; r dimes.

6. How many cents are there in n dollars? in $3x$ quarters? in $4n$ dimes?

7. Express in cents: a dollars and 10 cents; b quarters and 5 cents; c dimes and 2 cents; a dollars and 26 quarters, $3c$ dimes and 5 cents.

8. How many ounces are there in w pounds? in x pounds and 4 ounces?

9. Express in pounds: n ounces; w pounds and y ounces.

10. How many feet are there in n inches?

11. How many tons are there in w pounds?

12. How many feet are there in x miles?

13. How many feet are there in $2d$ yards?

14. How many dollars are there in n dimes?

15. How many quarters are there in a dollars?

16. Change into months: y years; t years and 3 months.

17. Express in minutes: t hours; n hours and 5 minutes.

18. Express in hours: a minutes; b days.

19. A boy's age was a years and 2 months. Express his age in months.

20. The length of a line was $3d$ feet and 5 inches. Express the length in inches.

21. Write a formula for changing feet to inches.

Ans. $i = 12f$.

22. Write a formula for changing inches to feet.

23. Write a formula for changing dollars to cents.

24. Write a formula for changing cents to dollars.

25. Write a formula for changing pounds to ounces.

26. An assembly hall has two sections, each of r rows having s seats in a row. Write a formula for the total number of seats (n) in the hall.

27. In an orchard there were a rows of trees, with n trees in each row. How many trees were planted in the orchard?

28. Water weighs 62.5 pounds per cubic foot. What is the pressure on the bottom of a rectangular tank which is filled to a depth of 5 feet, if the tank is b feet long and w feet wide?

29. A rule for making coffee is: "One spoonful of coffee for each person, and one for the pot." Change this rule to a formula by letting p represent the number of persons and s represent the number of spoonfuls of coffee.

30. A grocer bought a box of oranges containing n oranges. He sold them at c cents a dozen. If he paid d dollars for the box, what is his profit (p)?

31. The width of a rectangle was w. Its length was 3 larger. Write an expression for the perimeter of this rectangle.

32. The fire-insurance rate of a certain company, for each thousand dollars of insurance, was p dollars for three years. What is the insurance per year on $4000? on $6300?

30. Obtaining Formulas from Tables. The formulas which you have studied thus far were, in most cases, discovered very easily by a simple process of *reasoning*. This is true, for example, of such formulas as

$$A = lw, \qquad T = nc,$$
$$V = lwh, \qquad d = rt.$$

There are many formulas, however, which cannot be obtained exclusively by such a process. The following illustrative examples may serve to make this clear:

Illustrative Example 1. Suppose that a stone is dropped from a window which is 256 feet from the ground. How long will it take the stone to reach the ground?

Evidently no amount of mere thinking will enable you to get the result. How, then, is the answer to such a question to be found?

You can readily see that only a great many *experiments* would help us in the case of such a problem. Thus, if we could measure exactly the distance which a falling body covers in one second, in two seconds, and so on, we might obtain a rule that would apply in all such cases. As a matter of fact, such experiments were carried on by scientists. They found results very much like those shown in the following table:

Number of seconds =	1	2	3	4	5
Distance (in feet) =	16	64	144	256	400

It was discovered that this table can be replaced by a formula which, in a simplified form, becomes

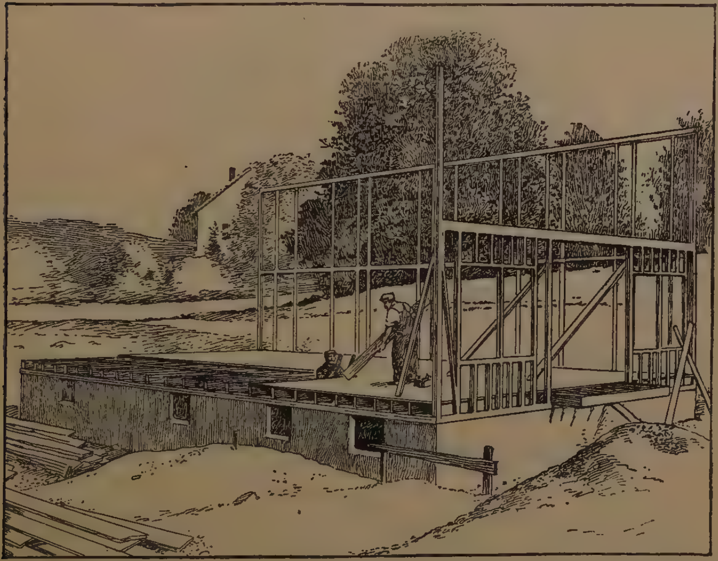
$$s = 16\,t^2.$$

This means that the distance (or *space*, in feet) which a falling body covers is 16 times the square of the *time* (in seconds). Do the numbers in the table agree with this rule? By using the above table, or the formula based on it, we find that a stone which is dropped from a height of 256 feet will reach the ground in *four* seconds.

NOTE. One of the most inventive men the world has ever seen was the famous Italian astronomer and mathematician Galileo (1564–1642). He was born in the ancient town of Pisa. As a student he became interested in the problem of falling bodies. There is a legend that one of his experiments consisted in dropping objects of varying weights from the famous Leaning Tower in his native city. It was through the pioneer work of Galileo, followed by that of other great thinkers, that the formulas of the laws which govern bodies that are in motion were finally discovered.

The Leaning Tower of Pisa, which is shown in the picture on page 66, was begun in the year 1174. It is 181 feet in height.

Illustrative Example 2. When it is necessary to build a bridge, or a tunnel, or a skyscraper, the engineers and contractors must be able to figure out, before beginning the work, exactly how strong each part of the structure must be in order that there may be no danger of a collapse. It must be known beforehand, for example, what is the *safe*



In the construction of an ordinary frame house many mathematical questions must be answered. Thus, all the wooden beams must be of the correct size, and they must be spaced and supported properly

load that steel beams or wooden beams of given sizes can carry. To answer such questions, engineers use *tables*, or formulas based on these tables. These tables and formulas can be obtained only by many experiments carried on in shops and laboratories.

For example, how many pounds can a rectangular beam of Georgia pine, of a given size, safely carry at its center, if the beam is supported at both ends? The formula is

$$w = \frac{500\,bd^2}{l},$$

where $w =$ the weight in pounds,
$b =$ the breadth of the beam in inches,
$d =$ the depth of the beam in inches,
and $l =$ the length of the beam in feet.

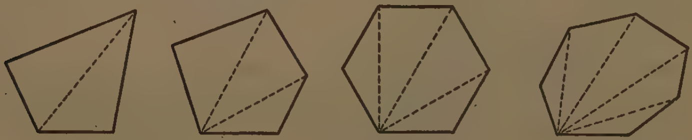
Thus, when $b = 6$ in., $d = 9$ in., and $l = 12$ ft., we have

$$w = \frac{500 \times 6 \times 81}{12} = 20{,}250 \text{ pounds.}$$

The following simple exercises will give you some practice in making tables, as well as formulas based on tables.

Preliminary Exercises.* *Making Tables and Formulas*

1. Draw a set of figures such as the following, the number of sides used in succession being 4, 5, 6, 7, and so on.



From one corner of each figure draw lines to all the other corners, as shown in the diagrams above. Such lines are often called *diagonals*. Complete the following table, which shows the number of diagonals that can be drawn in each of the figures.

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* It is suggested that these exercises be discussed informally. They constitute a development lesson.

Number of sides =	4	5	6	7	8	9
Number of diagonals =	1	2				

2. Observe that in this table we have two sets of numbers. These are arranged in pairs. That is, each number in the upper row has a corresponding number in the lower row. Now *compare* the numbers in the two rows. In Exercise 1 *each* number in the lower row is how much less than the corresponding number in the upper row?

3. Can you state the *relation* which you observed in the table of Exercise 1 in the form of a *rule*?

4. If the number of sides is represented by s, and the number of diagonals by d, show that the formula suggested by the table of Exercise 1 is $d = s - 3$. Explain.

5. Show that the formula mentioned in Exercise 4 could also be written in the form $s = d + 3$. Explain.

6. Using either of these formulas, find d when $s = 40$; when $s = 100$. Find s when $d = 17$; when $d = 25$.

7. To obtain these formulas, observe that the following four steps were used:

(1) *Making the table.*

(2) *Comparing the corresponding numbers in the table, and thus obtaining their relation.*

(3) *Expressing this relation as a rule.*

(4) *Writing the formula, or the shorthand form of the rule.*

8. Using the method suggested ~~above, make a~~ table showing how many *~~triangles are~~* formed in each of the figures on the preceding page. Using the letters s and t, write a formula based on this table.

Exercises. *Tables and Formulas*

REMEMBER THE KEY WORDS: **table, relation, rule, formula.**

1. A group of Boy Scouts started on a hike. The total distance they covered, from hour to hour, is shown in the following table:

Number of hours =	1	2	3	4	6	8
Distance in miles =	4	8	12			

Write the formula suggested by this table. (Use d and h.) By using this formula, find the missing numbers of the table.

In answering this question, some pupils wrote $d = 4\,h$, and others wrote $h = \frac{d}{4}$. Show that both statements are correct.

2. A pupil deposited money in the school bank each week. The total savings, in cents, at the end of each week are shown in the table up to the fifth week.

Number of weeks =	1	2	3	4	5	8	10	15
Savings =	8	16	24	32				

If the pupil continues to deposit money at the same rate, find the missing numbers in the table. Write a formula for the *savings* after any number of *weeks*. (Use s and w.)

3. A man earns 8 dollars a day. Make a table showing his wages from day to day, for one week. Write a formula showing the relation between *wages* and *days*.

4. If a pound of sugar costs 7 cents, make a table showing the total cost of 2 pounds, 3 pounds, and so on. Write a formula for the cost of *any* number of pounds. (Use c and p.)

5. A boys' camp was located 10 miles from the boys' home town. One day the boys started on a tramp. The table shows how far they were from their home, from hour to hour. Find a formula for their total distance from home after any number of hours. (Use d and h.) Find the missing numbers in the table.

Number of hours =	0	1	2	3	4	7	10
Distance from home =	10	13	16	19			

6. After a discussion of Exercise 5, some pupils suggested the formula $d = 10 + 3h$, while others expressed the relation between *distance* and *hours* by writing $h = \frac{d - 10}{3}$. Translate each formula into words and show that each expresses the relation given in the table.

7. Edwina Storm saved 60 cents during the summer. She put this money in the school bank at the beginning of the school year and then added 10 cents each week. Find a formula for her savings after any number of weeks. How much money did she have in the bank after 9 weeks?

8. An empty water tank was being filled at the rate of 10 gallons a minute. What is the formula for the number of gallons in the tank after any number of minutes?

9. In a large gasoline tank there were only 8 gallons of gasoline left on a certain day. It was then filled at the rate of 20 gallons a minute. Write a formula for the number of gallons in the tank after any number of minutes.

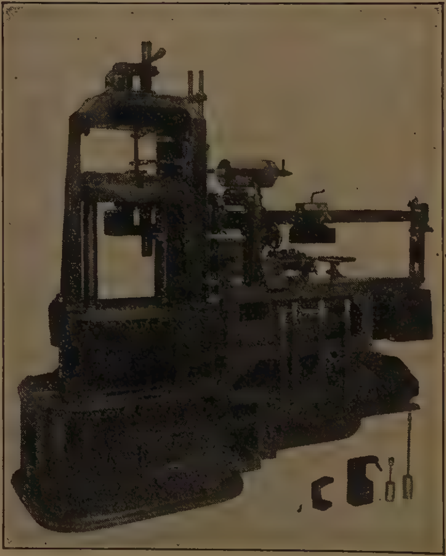
10. Henry Davis had saved 4 dollars during the summer. He found that he would have to spend 15 cents each week for school supplies. Make a formula showing how much money he had left after any number of weeks.

11. Henry was given a birthday present of 10 dollars. He decided to start a bank account with it. He earned 2 dollars each month, which he also saved. Write a formula expressing his savings (s) for any number of months (m).

*12. In a science laboratory a test was made to determine how much a certain wire would stretch if varying weights were attached to it.

Weight =	0	100	200	300	400
Length =	40	40.2	40.4	40.6	40.8

The table shows the results of the experiment. Thus, the original length of the wire was 40 inches. When a weight of 100 pounds was attached to it at one end and was allowed to stretch the wire vertically, the resulting length was found to be 40.2 inches, and so on. Write a formula connecting the weight and length.


A modern testing machine for determining the strength of steel bars and other structural materials

*13. A class in general science, while studying the subject of heat, wished to make a comparison of the centigrade

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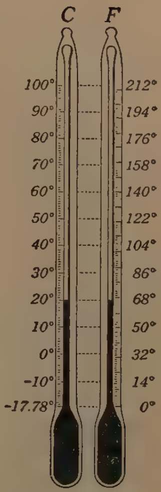
* These exercises may be postponed.



Comparison of the centigrade and Fahrenheit thermometers

and Fahrenheit thermometers. The pupils recorded a number of comparative readings on the two scales.

Some of these are shown in the table:

$C =$	0°	5°	10°	15°	20°
$F =$	32°	41°	50°	59°	68°

After studying these related numbers, the class found that the relation between the two scales could be expressed by either one of these formulas:

$$C = \frac{F - 32}{9} \times 5$$

and $$F = \frac{9\,C}{5} + 32.$$

Translate these two formulas.

Practice Exercises. *Tables and Formulas*

Study each of the tables in the following exercises. Try to discover the formulas based on each table. Find the missing numbers.

1.

If w is	7	9	18	30	40	75	100
then l is	11	13	22	34	44		

2.

When l is	20	37	40	71	92	100	250
then w is	17	34	37	68	89		

3.

If t is	3	5	9	15	20	55	60
then d is	60	100	180	300	400		

4.

If d is	30	45	60	120	180	210	300
then t is	10	15	20	40	60		

5.

If d is	16	24	32	60	80	160	200
then r is	4	6	8	15	20		

6.

If h is	3	4	5	6	7	10	20
then b is	7	9	11	13			

Solution. Observe that as the values of h increase by *one*, the corresponding values of b increase by *two*. If this relation had been true *from the beginning*, for each *unit increase* in h, it is clear that each value of b would be twice the corresponding value of h. That is, when $h = 3$, b would be 6. Instead, the value of b is 7, or one more than 6. Hence we infer that the required formula is $b = 2h + 1$.

Verify this formula by substituting in it the other pairs of numbers given in the table.

7.

If r is	3	4	5	6	7	10	15
then d is	10	13	16	19			

Show that the resulting formula is $d = 3r + 1$.

8.

When n is	5	7	10	15	17	20	50
then c is	19	27	39	59	67		

Solution. Observe that as n changes from 5 to 7, c changes from 19 to 27. That is, an increase of *two* in the value of n corresponds to an increase of *eight* in the value of c. Hence it would seem that an increase of *one* in the value of n would mean an increase of *four* in the corresponding value of c. If this relation had existed from the beginning for each *unit increase* in n, it is clear that when $n = 5$, the value of c would be 20. Instead, it is 19. Hence we infer that the required formula is $c = 4n - 1$. Verify this formula by substitution.

9.

If c is	2	5	7	10	15	20	50
then s is	13	25	33	45	65		

10.

If b is	9	12	15	24	30	45	108
then h is	5	6	7	10	12		

11.

If d is	2	5	10	12	20	70	100
then e is	7	19	39	47	79		

12.

When n is . . .	4	6	9	10	15	20	30
then s is	18	24	33	36	51		

13.

When a is . . .	6	10	14	20	38	40	200
then b is	2	4	6	9	18		

14.

When y is . . .	6	9	15	21	45	60	300
then x is	4	6	10	14	30		

15. Using the formula $s = 6w + 40$, make a table based on the following values of w: 0, 1, 3, 5, 6, 10, 20, 50.

16. Make a table based on the formula $t = 7r - 4$, using the following values of r: 1, 4, 5, 9, 12, 20, 50, 200.

17. Make a table based on the formula $y = \frac{1}{2}x + 4$, using the following values of x: 2, 6, 10, 14, 30, 40, 70.

31. The Table as an Expression of Relationships. We have seen in this chapter how mathematical relations, and hence formulas, may be obtained from a *table*.

Thus, consider the table at the top of page 76:

If $h =$	1	2	3	4	5	8	10
then $d =$	19	23	27	31	35		

An inspection of the table shows that the values of d and h *change together*, and that there is a definite relation between their corresponding values.

If we may assume that all the other corresponding values of d and h have the same relation, we can evidently express this relationship by means of the formula $d = 15 + 4h$. Once this formula is discovered, we can use it to extend the table indefinitely, or to find any desired missing value, such as those suggested above. Hence *a formula expresses a mathematical relation in a more general way than does a table.*

32. Summary. In this chapter you have learned

(1) How to obtain formulas from verbal statements.
(2) How to obtain formulas from tables.
(3) How to make tables based on formulas.

Test on Chapter III

1. Study the relation between the numbers in each of the following tables, and find the missing numbers.

(1)

If w is	3	4	5		17
then l is	1	2	3	10	

(2)

If t is	3	5	8	20	
then d is	9	15	24		72

(3)

If w is	1	2	5	7	
then s is	3	5	11		19

2. From each of the above tables obtain a formula which shows the relation of the letters used.

3. Using the formula $g = 200 + 25\,m$, make a table showing the values of g when $m = 0, 1, 2, 3, 4, 5$.

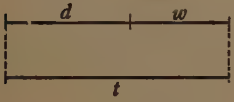
4. (1) Write a formula for the wages (w) of a man who earns 85¢ an hour for h hours.

(2) Write a formula for the total savings of a boy who starts with 60¢ and saves 15¢ each week. (Use s and w.)

(3) The cost (c) of sending a package by parcel post to a city in the second zone is 1¢ for each pound, plus 6¢. Write a formula for finding the cost, for any number of pounds (p).

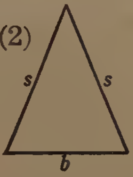
5. Write a formula suggested by each of the following diagrams:

(1)



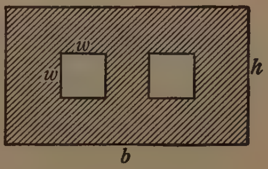
$t = ?$

(2)



Perimeter (p) = ?

(3)



Area (A) of shaded portion = ?

6. If the length of a rectangle is 8 and its width is w,

(1) Its perimeter equals - .

(2) Its area equals - .

(3) An area three times as large equals - - - - - - - - - - - - - - - .

(4) One fourth the area equals - .

7. Combine $20\,x^2y + 7\,xy^2 - x^2y + 10\,xy^2 + 5\,x^2y - 12\,xy^2$.

8. Combine $10\,a^3 + 4\,b^3 - 7\,a^3 + 8\,b^3 + 9\,a^3 - 10\,b^3 + a^3$.

9. Combine $9.2\,ab + 7.3\,ac + 12 - 7\,ab + 2.7\,ac - 10$.

10. Using the formula $R = \dfrac{3\,n}{ab + 4\,cd}$, find R if

$n = 100, \qquad a = 4, \qquad b = 5, \qquad c = 10, \qquad d = 2.$

CHAPTER IV

THE EQUATION

33. The Importance of Problems and Missing Numbers. In every numerical problem we are really trying to find the value of a missing number or of several missing numbers. There are many situations in everyday life which call for the finding of one or more missing numbers.

In the *home* it is of importance to know how much money can be set aside each week and month for food, clothing, and other necessary items. The *business man* and the *manufacturer* must find out their "overhead" expenses and know the cost of production and the probable margin of profit. The *builder* must determine the exact dimensions of all the parts of the house he is building, as well as the cost of each item in the contract. The *banker* must be able to tell what rate of interest he is obtaining from certain investments, and how to deal safely and profitably with deposits and other assets. The *engineer* must plan the dimensions and the necessary strength of the materials used in many structures, such as bridges, tunnels, skyscrapers, elevators, engines, motors, ships, and airplanes. The *government* must solve many financial problems of great importance to every citizen. *Inventors* and *scientists* are constantly trying to use our natural resources, as well as the great forces of heat, light, water, and electricity, for the improvement of living conditions. This requires the constant study of many problems.

Thus we see that not a person is living in the modern world whose welfare is not influenced very definitely by problems which someone must solve.

34. Solving Problems by the Equation Method. The simplest of these problems can be solved very quickly by ordinary arithmetic. When a problem is complicated, however, we usually need more powerful methods. The processes of algebra aid considerably in solving difficult problems. In particular, the great tool which algebra has furnished for finding missing numbers quickly and easily is the *equation method*. You will gradually become acquainted with this powerful method. We shall begin by comparing the way in which simple problems are solved by arithmetic and by algebra.

Illustrative Example 1. John's mother ordered five pounds of meat at the market. Her bill was 2 dollars. What was the price she had to pay per pound?

ARITHMETIC	ALGEBRA
Since five pounds cost 2 dollars, or 200 cents, and we wish to find the cost of one pound, we must divide 200 by 5. Hence the cost of one pound is 40 cents.	Let c represent the cost of one pound, in cents. Then $5c = 200$, and $c = 40$. (Why?) The statement $5c = 200$ is called an *equation*.

Illustrative Example 2. After traveling 4 hours in my automobile, I found that I had covered a distance of 84 miles. How fast had I traveled per hour?

ARITHMETIC. How do you solve this problem by arithmetic?

ALGEBRA. Let r represent the rate at which I traveled per hour.

Then $4r = 84$,

and $r = 21$. (Why?)

Again, the statement $4r = 84$ is called an *equation*.

Illustrative Example 3. James wishes to make a rectangular flower bed. He needs 40 square feet of land to plant the flowers he has bought. If the length of this plot is to be 8 feet, how wide must it be?

ARITHMETIC. How do you solve this problem by arithmetic?

ALGEBRA. Let w represent the width, in feet.

Then $8\,w = 40,$

and $w = 5.$ (Why?)

The statement $8\,w = 40$ is an *equation.*

Observe that the use of algebraic shorthand, in the solution of these problems, saves time and space. The following exercises will serve to give you some skill in writing equations which arise in the solution of simple problems.

Preliminary Exercises. *Making Equations*

In each problem represent the missing number by a letter. Then write the equation which may be used to solve the problem. Try to find the missing number in each case.

1. In 3 hours I covered a distance of 90 miles. What was my rate per hour? (Let $r =$ the rate.)

2. If I can travel at the rate of 20 miles an hour, how many hours will it take me to reach a town which is 160 miles away? (Let $h =$ the required number of hours.)

3. A grocer paid 32 dollars for 8 boxes of oranges. What was the cost per box? (Let $c =$ the cost per box, in dollars.)

4. If butter costs 50 cents a pound, how many pounds can I buy for 2 dollars?

5. After working 40 hours, my wages were $50. What was my hourly wage?

6. Ten years from now John's age will be 24 years. What is his present age?

7. Five years ago Henry was 8 years old. What is his present age?

8. If Carl's age were doubled, he would be 5 years older than his sister, who is 11 years old. How old is Carl?

9. A man left one half of his property to his wife. The remainder of the property was found to be \$8000. What was the value of his property?

10. If each side of a square were made 1 foot longer, the perimeter would be 64 feet. How long is one side?

1. Solving Equations by Inspection

35. Finding the Missing Number. In every equation it is necessary to find the value of a missing number, or of some *unknown quantity*. When we have found the value of this missing number, we are said to have *solved* the equation. It is one of the principal objects of algebra to teach us how to solve problems by means of equations. The following exercises will give you some practice in solving equations of the simplest kind.

Exercises. *Solving Easy Equations*

1. What number must be added to 6 to make 10? State the number for which the question mark stands in $6 + ? = 10$.

2. Find the value of the missing number in each of the following equations:

$9 + (\quad) = 12.$ $\qquad 7 + (\quad) = 30.$ $\qquad (\quad) + 3.1 = 9.6.$

$8 + (\quad) = 20.$ $\qquad (\quad) + 18 = 40.$ $\qquad (\quad) + 10 = 100.$

3. In all the preceding exercises, how did you find the missing number?

4. State the number denoted by the question mark in each case:

$4 + ? = 13.$ $? + 14 = 26.$ $60 = 40 + ?.$

5. State the number denoted by the question mark in each case:

2 times $? = 18.$ $\frac{1}{2}$ of $? = 6.$ $\frac{3}{5}$ of $? = 9.$

4 times $? = 40.$ $\frac{1}{3}$ of $? = 5.$ $\frac{4}{5}$ of $? = 8.$

6. Find the value of the missing number in each case:

$4(\quad) = 12.$ $2(\quad) = 7.$

$5(\quad) = 15.$ $3.1(\quad) = 9.3.$

7. Find the value of n in each case:

$n + 9 = 12.$ $n - 4 = 5.$ $\frac{1}{3}n = 2.$

$6n = 18.$ $n - 6 = 7.$ $\frac{1}{4}n = 10.$

8. In each of these tables find in the first column the numbers corresponding to those given in the second column:

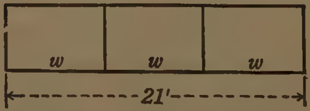
n	$n + 2$
—	7
—	8
—	12
—	40
—	9.7

n	$3n$
—	6
—	9
—	21
—	60
—	9.3

n	$n - 1$
—	7
—	9
—	15
—	400
—	9.3

n	$\frac{n}{4}$
—	2
—	5
—	10
—	80
—	2.3

9. Three boxes of equal widths were placed side by side, as shown in the figure. Find the value of w.

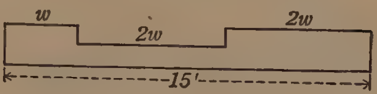


EXPLANATION. Since the three equal widths make up the total distance of 21 feet, we have the following equation:

$$3w = 21.$$

Hence $$w = 7. \text{ (Why?)}$$

10. In the diagram, find the value of w and of $2w$.



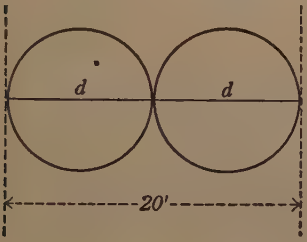
Equation. $5w = 15$.

Solution. $w =$ _______.

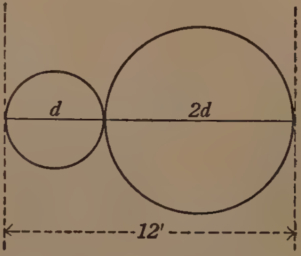
$2w =$ _______.

Find the diameters of these circles:

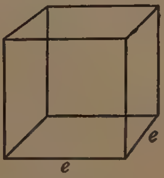
11.



12.



36. Comparing Evaluation and the Equation Method.* There is a very interesting and important relation between problems that are solved by *evaluation* and those which are solved by means of an *equation*. Study the following illustrative examples:



Example 1. Each boy in a shop class was asked to make a wire model of a cube of any convenient size. A record was kept of the amount of wire each boy used for his model, as shown in the following table:

	JOHN	HENRY	JAMES	CARL	FRED
$e =$	1″	$1\frac{1}{2}''$	2″	3″	5″
$t = 12e =$	12″	18″	24″	36″	60″

How many inches of wire did Fred use? Observe that in each case the value of t is obtained by evaluating the formula $t = 12e$.

---

* While it is strongly recommended that this section be studied at this point, it may be postponed at the discretion of the teacher.

Example 2. Suppose that each boy in the shop class mentioned above had been given a definite length of wire and had been told to make a model of a cube. The question would then be: "*How large* a model can I make with *this* piece of wire?"

	John	Henry	James	Carl	Fred
$t = 12\,e =$	24″	30″	36″	42″	60″
$e =$	2″				

Thus, if John received 24 inches of wire, he could make a two-inch cube, for if 12 equal pieces are necessary to make the model, *one* piece must be one twelfth of the total, or 2 inches. How large a model can each of the other boys make?

Observe that the missing value may be found in each case by solving an equation. Thus, in the case of John we have the equation

$$12\,e = 24,$$

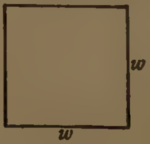
from which we obtain

$$e = 2.$$

You will see, then, that when a value of e was given, we found the corresponding value of t by *evaluation*. When a value of t was given, we found the corresponding value of e by *solving an equation*. In a certain sense, then, one process is the *reverse* of the other. The following exercises will serve to make you familiar with this important relation.

Exercises. *Evaluation and the Equation Method Compared*

1. A wire model of a square frame is to be made. Using the formula $p = 4\,w$, find the missing values in the following tables, all the dimensions being expressed in inches:



Evaluation

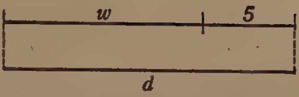
$w =$	3	5	9
$p = 4\,w =$			

Equation Method

$p = 4\,w =$	12	20	36
$w =$			

2. The accompanying diagram suggests the formula

$$d = w + 5.$$



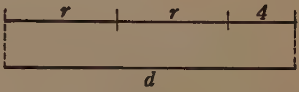
Using this formula, find the missing values in the following tables:

$w =$	2	10	16
$d = w + 5 =$			

$d = w + 5 =$	9	12	20
$w =$			

3. Write the formula suggested by the accompanying figure.

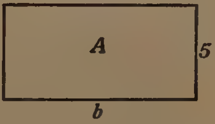
Using this formula, find the missing values in these tables:



$r =$	3	5	10
$d =$ ___ $=$			

$d =$ ___ $=$	12	20	64
$r =$			

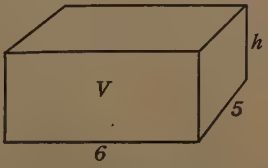
4. Using the formula based on the adjoining figure, find the missing values in the following tables, all the dimensions being in feet:



$b =$	8	10	12
$A = 5b =$			

$A = 5b =$	20	35	60
$b =$			

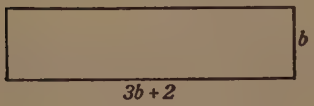
5. Write the formula for the volume of the rectangular solid shown in the diagram and thus find the missing values in the following tables:



$h =$	4	7	10
$V =$ ___ $=$			

$V =$ ___ $=$	60	90	100
$h =$			

6. Find the missing values in these tables, if p represents the perimeter of the figure below:



$b =$	4	7	10
$p = \_\_\_ =$			

$p = \_\_\_ =$	20	36	60
$b =$			
$3b + 2 =$			

2. Solving Equations by Rule

37. Importance of Skill in Solving Equations. Since equations are of very great assistance in solving mathematical problems, it is necessary for every student of algebra to develop skill in the use of this equation method.

In the following pages you will study certain explanations and rules which will enable you to solve equations quickly and accurately.

38. Equality. *An equality is a statement showing that two expressions stand for the same number.* Thus, the following statements represent different types of equalities.

1. $8 = 8$.
2. $4 + 2 = 6$.
3. $4w = 12$.
4. $7x + 2 = 9$.
5. $a + b = c$.
6. $A = lw$.

39. Identity. *An identity is an equality that states a fact.*

Thus, $4 + 2 = 6$ is an identity.

Similarly, $2n + 4n = 6n$ is an identity.

Also, $a + b = b + a$ is an identity.

For, whatever value is given to the letters which appear in either of these equalities, the statement remains an equality.

Hence, an equality which is true for all values of the letters which appear in it is an identity.

40. Equation. *An equation is an equality that asks a question.*

Thus, $5n + 2 = 17$ is an equation. It is really a short way of writing the question: "Five times *which* number, when two is added, will give the total result 17?"

In other words, this equation suggests a *particular* missing number which is to be found. Hence such an equality is true only if we replace the letter in the equation by the correct value.

Obviously, *it is the purpose of an equation to enable us to find quickly some missing number* (*or numbers*) *in a given problem.*

41. Root. The correct value of the missing number which is required in a given equation is called the *root* of the equation. The process of finding the root of an equation is often called the *solution* of the equation.

Thus, the solution of the equation $5n + 2 = 17$ shows that 3 is the root.

42. Checking a Solution. In order to make sure that the correct value of the missing number has been found, we must *check* each result. This is done by substituting the value found for the missing number in the given equation. If we obtain a true statement, or an identity, we know that the solution is correct.

Thus, the root of the equation $4w + 5 = 17$ is 3, for by substitution we find that

$$4 \times 3 + 5 = 17,$$

from which we have $$12 + 5 = 17,$$

or $$17 = 17.$$

The correct value of the root is said to *satisfy* the equation.

There are equations which have *more* than one root. For example, the equation $x^2 - 5x + 6 = 0$ has two roots, 2 and 3, as may be seen by substituting each of these values for x.

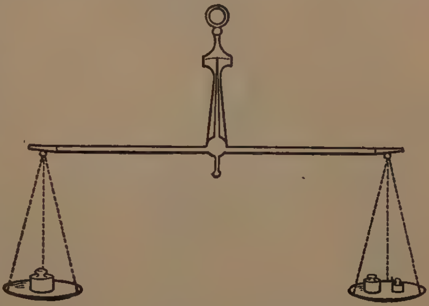
Practice Exercises. *Checking Solutions*

Copy the numbers of the following examples. In the case of each equation, check by substitution the indicated root. If the root checks, write the word "True"; if not, write "False."

1. $4a = 60.$ $\qquad a = 15.$
2. $7w + 1 = 15.$ $\qquad w = 2.$
3. $9h - 1 = 26.$ $\qquad h = 4.$
4. $4n + 7 - 2n = 15.$ $\qquad n = 2.$
5. $7x + 4 = 2x + 9.$ $\qquad x = 1.$
6. $\frac{2}{3}n + 6 = 10.$ $\qquad n = 6.$
7. $9 - 2a = 3.$ $\qquad a = 3.$
8. $25 - 3h = 40 - 6h.$ $\qquad h = 5.$
9. $4n + 1 = 17.$ $\qquad n = 6.$
10. $3d - 5 = 7.$ $\qquad d = 2.$
11. $a - 5 = 8.$ $\qquad a = 13.$
12. $x + 10 = 24.$ $\qquad x = 6.$
13. $40r - 10 = 7r + 30.$ $\qquad r = 4.$
14. $6 - a = 12 - 2a.$ $\qquad a = 4.$
15. $7c - 5 = 4c + 4.$ $\qquad c = 3.$

43. Axioms and the Rule of Equality. In the preceding pages you have solved equations by *inspection.* But just as your work in arithmetic was made more definite by learning some important rules, you will find it very helpful to solve equations by applying a few simple principles, often called *axioms.* A knowledge of these will assist greatly in planning the solution of equations and in obtaining the required results more rapidly.

The two expressions which are written on both sides of the equality sign in an equation are often called its *members*. The left side of an equation is called its *first member*, while the right side is called the *second member*. The two members of an equation may be compared to equal weights which are placed in the scale pans of an ordinary balance. So long as these loads remain equal, they balance each other. If you change one of these weights, you must make a corresponding change in the other weight. Otherwise there will no longer be a balance. Keeping this illustration in mind, you will readily be able to understand the following four numerical statements:



	ADDITION	SUBTRACTION	MULTIPLICATION	DIVISION
GIVEN EQUALITY:	8 = 8	8 = 8	8 = 8	8 = 8
	2 = 2	2 = 2	2 = 2	2 = 2
RESULTING EQUALITY:	10 = 10	6 = 6	16 = 16	4 = 4

Thus, if you have two equal numbers, as is the case in the statement 8 = 8, you can *add* 2 to both numbers; or you can *subtract* 2 from both numbers; or you can *multiply* both numbers by 2; or you can *divide* both numbers by 2. In each case you obtain another equality, such as 10 = 10, or 6 = 6, and so on.

For example, suppose that each of two boys deposits 8 dollars in the school bank, and that later each withdraws 2 dollars. Then each boy has 6 dollars left in the bank. Similarly, it is easy to suggest suitable illustrations for the other three statements.

Hence we have the following four simple principles, or *axioms*:

1. *Axiom of Addition. If equals are added to equals, the sums are equal.*
2. *Axiom of Subtraction. If equals are subtracted from equals, the differences are equal.*
3. *Axiom of Multiplication. If equals are multiplied by equals, the products are equal.*
4. *Axiom of Division.*[*] *If equals are divided by equals, the quotients are equal.*

These four statements may be combined into a single statement, called the Rule of Equality:

> *If equals are increased, or decreased, or multiplied, or divided by equals, the results are equal.*

This rule is sometimes stated in the following form: *Whatever you do to one side of an equation you must also do to the other side.*

We shall now use the Rule of Equality in solving various types of equations.

44. Equations of the First Type. (Using Division.) Study the following illustrative examples:

Illustrative Example 1. Solve the equation $5n = 20$.

The given equation tells us that a certain number, when multiplied by 5, gives 20.

Solution.	(1) $5n = 20$	Given equation.
	(2) $\underline{\ \ 5 = \ 5}$	Identity.
	(3) $\ \ n = \ 4$	Axiom 4.

Here we have divided each member by 5, thus applying Axiom 4.

In brief form, the solution may be written:

(1) $5n = 20.$	Given.
(2) $n = 4.$	D_5.

The abbreviation D_5 means that we have divided each member by 5.

---

[*] The case of division by zero is excluded, for reasons which will be considered in a later chapter.

Check. The answer is 4, for if we substitute 4 for n in the given equation we have

$$5 \times 4 = 20,$$

or

$$20 = 20.$$

Illustrative Example 2. Solve the equation $4.1\,a = 82$.

Solution.	(1) $4.1\,a = 82$.	Given.
	(2) $a = 20$.	$D_{4.1}$.

Check. $4.1 \times 20 = 82$,

or $82 = 82$.

$$\begin{array}{r} 4.1 \\ \underline{20} \\ 82.0 \end{array}$$

Exercises. *Equations of the First Type*

Numbers 1 to 10, oral

Solve the following equations and check each result:

1. $4\,a = 12$.
2. $7\,c = 21$.
3. $10\,d = 100$.
4. $20\,h = 90$.
5. $9\,x = 9$.
6. $12\,n = 84$.
7. $6\,p = 1200$.
8. $5\,d = 600$.
9. $8\,r = 17$.
10. $2\,p = 21$.
11. $4\,a + 3\,a = 21$.
12. $7\,d - 2\,d = 40$.
13. $20\,r - 4\,r = 32$.
14. $10\,w + w = 44$.
15. $4\,a = 5$.
16. $7\,w = 0$.
17. $3\,b = 2$.
18. $1 = 2\,d$.
19. $5\,x = 4$.
20. $6\,h = 2.4$.
21. $2.3\,x = 69$.

Study each of the following problems. Find out what is required. Try to state the problem in the form of an equation, and solve it.

22. In 5 hours an automobile covered a distance of 150 miles. What was its average rate per hour? A $\therefore 5r = 150$

SUGGESTION. If the average rate is called r, we have $5\,r = 150$. Explain.

$S = \frac{60}{4}$

23. If the perimeter of a square is to be 60 feet, how long must one side be?

24. A salesman found at the end of a year that his total commission exactly equaled his regular salary. His total income was $6400. What was his salary? (Let $c =$ the commission. Then $2c = 6400$. Explain.)

25. In 10 years a man saved $8970. How much did he save annually?

26. If you wish to reach in 5 hours a city which is 120 miles from your home, at what rate must you travel?

27. Two business partners wished to divide a profit of $12,000. If one contributed twice as much capital as the other, how much should each receive?

28. There were 100 shares in a business owned by three partners. A owned 52 shares, B owned 30 shares, and C owned the rest. A profit of $8400 is to be divided among them. How much should each receive?

29. A liter contains 61 cubic inches, very nearly. How many liters are there in 10 gallons, if one gallon contains 231 cubic inches?

Let l represent the number of liters. Then $61\,l = 10 \times 231$. Explain.

30. The light which leaves the sun reaches the earth in about $8\frac{1}{3}$ minutes. The sun's distance from the earth is about 93,000,000 miles. How far does light travel in a second?

45. Equations of the Second Type. (Using Multiplication.) Study the following illustrative example:

Illustrative Example. Solve the equation $\frac{n}{3} = 6$.

The given equation states that one third of a certain number is 6.

Solution.	(1) $\frac{n}{3} = 6$	Given equation.
	(2) $\underline{3 = 3}$	Identity.
	(3) $n = 18$	Axiom 3.

Here we have multiplied each member of the given equation by 3, thus applying Axiom 3.

In brief form, this may be written:

(1) $\frac{n}{3} = 6$. | Given.
(2) $n = 18$. | M_3.

The abbreviation M_3 means that we have multiplied each member by 3.

NOTE. Remember that $\frac{n}{3}$ has the same meaning as $\frac{1}{3}n$.

Check. $\frac{18}{3} = 6$,

or $6 = 6$.

Exercises. *Equations of the Second Type*

Numbers 1 to 10, oral

Solve the following equations and check each result:

1. $\frac{n}{2} = 7$.
2. $\frac{d}{3} = 9$.
3. $\frac{x}{10} = 2$.
4. $\frac{a}{5} = 20$.
5. $50 = \frac{l}{10}$.
6. $19 = \frac{1}{4}x$.
7. $2 = \frac{1}{9}h$.
8. $\frac{1}{100}c = 3$.
9. $\frac{1}{100}p = 21$.
10. $\frac{1}{7}n = 1000$.
11. $\frac{n}{5} = 9$.
12. $\frac{n}{19} = 20$.
13. $100 = \frac{x}{7}$.
14. $\frac{x}{6} = 0$.
15. $\frac{a}{10} = 1\frac{1}{2}$.
16. $6.4 = \frac{h}{3}$.
17. $\frac{p}{100} = 4.72$.
18. $\frac{p}{100} = 17.93$.
19. $2\frac{1}{5} = \frac{h}{5}$.
20. $\frac{h}{8} = 5\frac{3}{4}$.
21. $\frac{x}{8} = 2\frac{1}{3}$.

Solve each of the following problems by means of an equation:

22. I decided to save one fifth of my income during a certain month. My savings during that month were \$18.20. What was my income?

23. In a game Henry scored 21 points, which were one fourth of all the points scored. What was the total score in the game?

24. James is one fifth as old as his father. James is 8 years old. What is his father's age?

25. After completing one half of her lesson, Martha had worked 10 examples. How many examples had been assigned?

26. After reading 30 pages of a book, I found that I had read only one tenth of the total number of pages. How many pages were there in the book?

46. Equations of the Third Type. (Using Subtraction.) Study the following illustrative examples:

Illustrative Example 1. Solve the equation $n + 3 = 17$.

Solution.

$$\begin{array}{lr|l} (1) & n + 3 = 17 & \text{Given equation.} \\ (2) & 3 = \ \ 3 & \text{Identity.} \\ \cline{1-2} (3) & n = 14 & \text{Axiom 2.} \end{array}$$

Here we have subtracted 3 from each member of the original equation, thus applying Axiom 2.

In brief form, this solution may be written:

$$\begin{array}{lr|l} (1) & n + 3 = 17. & \text{Given.} \\ (2) & n = 14. & S_3. \end{array}$$

S_3 means that 3 was subtracted from each member of the original equation.

Check. $$14 + 3 = 17,$$

or $$17 = 17.$$

Illustrative Example 2. Solve the equation $x + 4.1 = 9.7$.

Solution.

$$\begin{array}{lr|l} (1) & x + 4.1 = 9.7. & \text{Given.} \\ (2) & x = 5.6. & S_{4.1}. \end{array}$$

Check. $$5.6 + 4.1 = 9.7,$$

or $$9.7 = 9.7.$$

Exercises. *Equations of the Third Type*

Numbers 1 to 10, oral

Solve the following equations and check each result:

1. $a + 6 = 10$.
2. $n + 9 = 20$.
3. $w + 100 = 300$.
4. $d + 90 = 91$.
5. $n + 7.1 = 9.1$.
6. $x + 1\frac{1}{2} = 6\frac{1}{2}$.
7. $10 + x = 19$.
8. $20 + d = 60$.
9. $50 + n = 52$.
10. $100 + x = 101$.
11. $n + 4 = 6$.
12. $d + 10 = 17\frac{1}{2}$.
13. $x + \frac{1}{2} = 7\frac{1}{2}$.
14. $a + .1 = 9.1$.
15. $h + 17 = 30$.
16. $b + 1\frac{1}{3} = 6\frac{1}{3}$.
17. $c + 3.2 = 9.2$.
18. $x + 500 = 600$.
19. $t + 15\frac{1}{3} = 17$.
20. $r + 7.4 = 10$.

Study the following problems. Find out in each case what is required. Write the equation which represents the problem, and solve it.

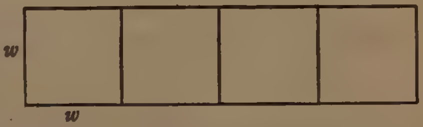
21. After receiving a salary increase of $110, a clerk's salary was $1400. What was the original salary?

22. A firm announced an increase of 5¢ per pound in the price of beef. The new price was 40¢. What was the original price?

23. By increasing the speed of his car 10 miles per hour, a chauffeur was able to cover a distance of 35 miles in one hour. What was his original speed?

24. The total length of wire needed to make a certain wire model of a cube was found to be 48 inches. How long was the edge of the cube?

25. A wire frame consisting of four equal panels, as shown in the diagram, required 52 inches of wire. How long was one side of each square?



26. The hourly rates of three trains, in miles, are d, $2d$, and $3d$ respectively. If the total distance covered by the trains in 4 hours was 480 miles, what was the rate of each train?

27. The annual incomes of four members of a family are i, $3i$, $4i$, and $10i$ respectively. Their combined annual income was found to be $5400. How much did each of the four members earn per year?

28. Three men, A, B, and C, go into business together. B contributes twice as much capital as A, and C contributes three times as much capital as A. If the profit during a certain year was $12,000, what was each man's share?

47. Equation of the Fourth Type. (Using Addition.) Study the following illustrative example:

Illustrative Example. Solve the equation $n - 4 = 5$.

Solution. This equation tells us that a certain number, when diminished by 4, gives 5.

(Remember that $10 - 4 + 4 = 10$. Similarly, $n - 4 + 4 = n$.)

(1) $n - 4 = 5$	Given equation.
(2) $\underline{\quad 4 = 4}$	Identity.
(3) $n = 9$	Axiom 1.

Here we have added 4 to each member of the original equation, thus applying Axiom 1. In brief form, this solution may be written:

(1) $n - 4 = 5$.	Given.
(2) $n = 9$.	A_4.

(What is the meaning of A_4?)

Check. $9 - 4 = 5$,

or $5 = 5$.

Exercises. *Equations of the Fourth Type*

Numbers 1 to 10, oral

Solve the following equations and check each result:

1. $a - 2 = 7$.
2. $d - 7 = 10$.
3. $w - 6 = 100$.
4. $n - 17 = 1$.
5. $h - 6 = 49$.
6. $x - \frac{1}{2} = 7$.
7. $c - 1\frac{1}{3} = 2$.
8. $n - 1.1 = .9$.
9. $r - 2 = 7\frac{1}{2}$.
10. $x - 10 = 14.5$.
11. $x - 4 = 6$.
12. $n - 10 = 20$.
13. $d - 12 = 8$.
14. $w - 20 = 20$.
15. $x - 2\frac{1}{2} = 4\frac{1}{2}$.
16. $d - 3.9 = 6.1$.
17. $w - 100 = 0$.
18. $n - 5\frac{2}{3} = 8\frac{2}{3}$.
19. $r - 4\frac{5}{8} = 3\frac{7}{8}$.
20. $n - 8.7 = 15.2$.

Try to solve each of the following problems by the equation method:

21. I am thinking of a number which when diminished by 8 is equal to 20. What is the number?

22. After paying \$2.50, I still had \$7. How many dollars did I have at first?

23. After cutting off a piece of ribbon 7 inches long, I had 29 inches of ribbon left. What was the original length of the ribbon?

24. What number, when decreased by $29\frac{4}{5}$, becomes $31\frac{1}{5}$?

25. After traveling a distance of 49.1 miles, I still had to go 23.9 miles to reach my destination. What was the total distance I had planned to cover?

48. Summary. We have now studied four simple types of equations. The solution of each requires a special process. Complete the following table:

The *addition type*, such as $n + 3 = 7$, is solved by *subtraction*.
The *subtraction type*, such as $n - 2 = 6$, is solved by ______.
The *multiplication type*, such as $4\ n = 12$, is solved by ______.
The *division type*, such as $\frac{n}{4} = 2$, is solved by __________.

What relation exists, therefore, between the operation shown in the *given* equation and the operation which is used in *solving* it?

Timed Practice Test

Time, 2 minutes

1. $4\ a = 12.$
2. $7\ w = 21.$
3. $8\ h = 120.$
4. $5\ d = 600.$
5. $9\ x = 9.$
6. $10\ x = 20.$
7. $12\ x = 60.$
8. $15\ n = 0.$
9. $105\ p = 31{,}500.$
10. $6\ p = 1200.$
11. $2\ a = 40.$
12. $3\ b = 90.$
13. $4\ n = 28.$
14. $4.1\ s = 8.2.$
15. $33\ d = 132.$
16. $14\ n = 56.$
17. $9.3\ k = 27.9.$
18. $4\ w = 100.$
19. $500\ c = 15{,}000.$
20. $25\ n = 25.$

Timed Practice Test

Time, 2 minutes

1. $n + 4 = 9.$
2. $n + 7 = 12.$
3. $w + 4 = 6.$
4. $d + 10 = 20.$
5. $a + .1 = 9.1.$
6. $x + \frac{1}{2} = 7\frac{1}{2}.$
7. $h + 17 = 30.$
8. $b + 1\frac{1}{3} = 6\frac{1}{3}.$
9. $n + 3.2 = 9.2.$
10. $x + 500 = 600.$
11. $x + 15\frac{1}{3} = 17.$
12. $r + 7.4 = 10.$

13. $n + \frac{1}{3} = 2.$ 15. $n + 7\frac{1}{3} = 20.$ 17. $n + 3.15 = 8.45.$
14. $n + 2\frac{1}{2} = 7.$ 16. $n + 2.1 = 9.1.$ 18. $n + 500 = 1500.$
19. $n + 60\frac{1}{2} = 90\frac{1}{2}.$ 20. $n + 18.5 = 20.6.$

Timed Practice Test

Time, 2 minutes

1. $n - 1 = 8.$ 4. $n - 2\frac{1}{2} = 3\frac{1}{3}.$ 7. $x - 12 = 8.$
2. $n - 2 = 10.$ 5. $n - 4 = 6.$ 8. $x - 20 = 20.$
3. $n - 6 = 4.$ 6. $x - 10 = 0.$ 9. $x - 100 = 1.$
10. $n - 97.43 = 146.27.$ 16. $y - 1 = 1.$
11. $n - 200 = \frac{1}{3}.$ 17. $n - \frac{1}{2} = \frac{1}{2}.$
12. $n - 6.1 = 8.4.$ 18. $n - 9.4 = 3.5.$
13. $x - 3.9 = 6.1.$ 19. $n - 16.4 = 2.$
14. $n - 2\frac{1}{5} = 7.$ 20. $n - 200 = 300.$
15. $y - 3.1 = 9.$ 6.9 21. $x - 400 = 310.$

49. Equations Solved by Using Two Processes. As a rule, more than one operation is required to solve the equations which arise in dealing with actual problems. We shall begin by considering equations which may be solved by using two operations, or two "steps." The explanations which follow may easily be extended to the case of equations requiring for their solution three or more steps. Study the following illustrative examples:

Illustrative Example 1. Solve the equation $4n + 5 = 17.$ (State the equation in words before solving.)

Solution. Subtracting 5 from each of these equals, we have

$$4n = 12. \qquad \text{[By Axiom 2]}$$

Dividing each member by 4, we have

$$n = 3. \qquad \text{[By Axiom 4]}$$

In brief form, the solution may be written:

(1) $4n + 5 = 17.$	Given.
(2) $4n = 12.$	S_5.
(3) $n = 3.$	D_4.

Check. By substituting this result in the original equation, we have

$$4 \times 3 + 5 = 17,$$

or $$12 + 5 = 17,$$

or $$17 = 17.$$

Illustrative Example 2. Solve the equation $4a - 3 = 9$. (State the equation in words before solving.)

Solution.

(1) $4a - 3 = 9.$	Given.
(2) $4a = 12.$	A_3.
(3) $a = 3.$	D_4.

Check. $$4 \times 3 - 3 = 9,$$

or $$12 - 3 = 9,$$

or $$9 = 9.$$

Illustrative Example 3. Solve the equation $\frac{n}{5} + 2 = 6$. (State the equation in words before solving.)

Solution.

(1) $\frac{n}{5} + 2 = 6.$	Given.
(2) $\frac{n}{5} = 4.$	S_2.
(3) $n = 20.$	M_5.

Check. $$\tfrac{20}{5} + 2 = 6,$$

or $$4 + 2 = 6,$$

or $$6 = 6.$$

Illustrative Example 4. Solve the equation $\frac{2}{3}n = 8$. (State the equation in words before solving.)

Solution by Inspection. Since $\frac{2}{3}$ of a certain number is 8, $\frac{1}{3}$ of that number must be 4, and hence the required number is 3 × 4, or 12.

Solution by Rule.

FIRST METHOD.

(1) $\frac{2}{3}n = 8.$	Given.
(2) $\frac{1}{3}n = 4.$	D_2.
(3) $n = 12.$	M_3.

Second Method. We may divide each member at once by the coefficient of n, namely $\frac{2}{3}$. Remember that dividing 8 by $\frac{2}{3}$ is the same as multiplying 8 by $\frac{3}{2}$, the reciprocal of $\frac{2}{3}$. Hence we may write the solution as follows:

(1) $\frac{2}{3}n = 8$.	Given.
(2) $n = 12$.	Dividing by $\frac{2}{3}$ (Ax. 4).

Check. $\frac{2}{3} \times 12 = 8$,

or $8 = 8$.

Note. *Remember that* $\frac{2}{3}n$ has the same meaning as $\frac{2n}{3}$. Verify this statement by substitution. Similarly, $\frac{4}{5}a = \frac{4a}{5}$, and $\frac{3}{100}p = \frac{3p}{100}$.

Exercises. *Equations Solved by Using Two Processes*

Numbers 1 to 10, oral

Observe that in all the equations which you have solved thus far, you used the reverse of the operations which appeared in the original equation.

Solve the following equations and check each solution:

1. $2n + 1 = 11$.
2. $3n + 3 = 15$.
3. $4n + 6 = 66$.
4. $6n + 15 = 45$.
5. $4d - 1 = 11$.
6. $7h - 2 = 26$.
7. $10l - 7 = 13$.
8. $8r - \frac{1}{2} = 15\frac{1}{2}$.
9. $\frac{d}{4} + 3 = 6$.
10. $\frac{n}{5} + 7 = 9$.
11. $\frac{1}{10}h + \frac{1}{3} = \frac{1}{2}$.
12. $\frac{1}{4}p + 4.1 = 8.1$.
13. $\frac{n}{2} - 3 = 5$.
14. $\frac{w}{3} - 6 = 0$.
15. $\frac{1}{4}b - \frac{1}{3} = 4\frac{2}{3}$.
16. $\frac{1}{5}c - 7.1 = 8.9$.
17. $\frac{2}{3}a = 6$.
18. $\frac{3}{5}b = 9$.
19. $\frac{4}{7}d = 8$.
20. $\frac{3}{100}c = 12$.
21. $\frac{5n}{7} = 10$.
22. $\frac{6n}{100} = 0$.
23. $\frac{5}{8}x = 15$.
24. $2\frac{1}{2}x = 5$.
25. $3\frac{1}{3}d = 20$.
26. $4\frac{1}{5}n = 42$.
27. $12\frac{1}{2}d = 100$.
28. $\frac{7}{8}n = 14$.
29. $3\frac{1}{7}d = 22$.
30. $.04x = 8$.

31. $.05\,c = 40$.
32. $.06\,p = 12$.
33. $.03\,x = 24.66$.
34. $\frac{2}{3}\,d + 4 = 12$.
35. $\frac{3}{5}\,n - 4 = 2$.
36. $\frac{7}{8}\,x + 4 = 11$.
37. $\frac{8}{9}\,w - 1 = 15$.
38. $4\frac{1}{3}\,h + 3 = 16$.
39. $.1\,d = 1$.
40. $.3\,n = 6$.
41. $4\,n + 80 = 90$.
42. $6\,w - 2.6 = 19$.
43. $9\,a + \frac{1}{2} = 20\frac{1}{2}$.
44. $8\,d - 3 = 41$.
45. $\frac{n}{6} + 900 = 1000$.
46. $\frac{1}{7}\,d - 300 = 1$.
47. $\frac{15}{16}\,x + 3 = 18$.
48. $\frac{9}{10}\,r - 1\frac{1}{2} = 7\frac{1}{2}$.
49. $4\frac{1}{5}\,r + 9 = 51$.
50. $5\frac{1}{6}x - 30.4 = 31.6$.

Timed Practice Test

Time, 5 minutes

1. $4\,n + 3 = 15$.
2. $6\,n + 2 = 20$.
3. $5\,n + 7 = 27$.
4. $7\,n + 5 = 40$.
5. $2\,n + 1 = 3$.
6. $10\,x + 7 = 57$.
7. $11\,x + 4 = 70$.
8. $20\,x + 9 = 69$.
9. $17\,n + 2 = 53$.
10. $100\,n + 500 = 700$.
11. $8\,x + 7 = 23$.
12. $9\,x + 5 = 50$.
13. $12\,x + 4 = 76$.
14. $7\,x + 3 = 73$.
15. $4\,n + 2 = 9$.
16. $3\,n + 7 = 10$.
17. $5\,n + 2.1 = 12.1$.
18. $2.1\,x + 6 = 12.3$.
19. $3\,x + 1 = 2$.
20. $4\,x + 3 = 5$.

Timed Practice Test

Time, 5 minutes

1. $5\,n - 9 = 1$.
2. $6\,w - 4 = 8$.
3. $7\,w - 10 = 4$.
4. $8\,n - 1 = 15$.
5. $5\,n - 2 = 8$.
6. $3\,n - 15 = 6$.
7. $2\,n - 7 = 13$.
8. $5\,x - 6 = 24$.
9. $6\,x - 1 = 5$.
10. $2\,x - 5 = 5$.
11. $3\,y - 6 = 0$.
12. $5\,y - 1 = 14$.
13. $7\,y - 4 = 3$.
14. $10\,y - 7 = 13$.
15. $12\,y - 7 = 53$.
16. $20\,x - 1 = 59$.
17. $2\,x - 5.1 = 7.9$.
18. $3\,x - 4.3 = 8$.
19. $4\,x - 3 = 7$.
20. $7\,x - 4 = 5$.

Timed Practice Test

Time, 5 minutes

1. $\frac{n}{3}+1=5.$
2. $\frac{n}{2}+5=7.$
3. $\frac{n}{5}+1=8.$
4. $\frac{n}{3}+4=6.$
5. $\frac{n}{4}-1=6.$
6. $\frac{x}{7}-2=3.$
7. $\frac{x}{9}-1=10.$
8. $\frac{n}{2}+7=9.$
9. $\frac{w}{6}+5=8.$
10. $\frac{d}{10}+100=150.$
11. $\frac{5}{6}n=10.$
12. $\frac{5w}{9}=100.$
13. $\frac{3}{2}x=12.$
14. $\frac{2}{3}x+1=5.$
15. $\frac{3}{5}x+2=8.$
16. $\frac{3}{4}x-1=5.$
17. $\frac{5}{8}x-2=3.$
18. $\frac{8w}{5}=16.$
19. $\frac{m}{45}+6\frac{1}{3}=9\frac{1}{3}.$
20. $.06p=120.$

50. Composite Types of Equations. The solution of a given equation may, of course, require the use of more than two steps. In such a case we may apply the fundamental axioms repeatedly until we obtain the required missing number.

Illustrative Example. Solve the equation $6a+3=4a+5$.

Solution.	(1)	$6a+3=4a+5.$	Given.
	(2)	$6a=4a+2.$	S_3 (Ax. 2).
	(3)	$2a=2.$	S_{4a} (Ax. 2).
	(4)	$a=1.$	D_2 (Ax. 4).

Check. $6\times1+3=4\times1+5,$

or $6+3=4+5,$

or $9=9.$

Exercises. *Composite Types of Equations*

1. $14a+60-2a=72.$
2. $3w+15+2w=18+4w.$
3. $6n+4-n=n+20.$
4. $12+n=7n+6.$
5. $8+2h=7h+3.$
6. $4x+4+6x=9x+6.$

7. $2x + 20 + 33x = 40 + 15x.$
8. $12d + 4 - 3d = 39 + 2d.$
9. $7n = 0.$
10. $100 + n = 9n + 20.$
11. $3x - 6 = 0.$
12. $\frac{1}{2}h = 0.$
13. $8x - 7 + 2x = 3.$
14. $17r - 34 = 0.$
15. $10n + 8 + 6n + 6 + 2n + 2 = 6n + 40.$
16. $4x + 8 + 2x + 7 - x = 15 + x + 10 + 2x.$
17. $9n + 7 + 3n + 5 = 8n + 22 - 2n - 10.$
18. $8n + 3 = 10n - 4.$
19. $9n + 7 - 6n = 12n + 4.$
20. $14n + 8 - 7n = 21n - 2.$
21. $3a + 4 = a + 6.$
22. $4b + 9 = 2b + 15.$
23. $7w - 6 = 2w + 4.$
24. $8r - 7 = 5r + 14.$
25. $9l + 2 = 8 + 7l.$
26. $8w + 6 = 9 + 5w.$
27. $4w + 6 = 5w + 3.$
28. $9a + 10 = 7a + 16.$
29. $10b + 4 = 12b + 2.$
30. $9w - 4 = 6 + 4w.$
31. $8a + 6 + 5a + 3 = 6a + 23.$
32. $2.8a + 6 + 9.2a = 12.$
33. $8x + 4 + 7x = 9x + 10.$
34. $9d - 7 + 2d = 14 + 4d.$
35. $8l + 5 - 4l = 23 - 5l.$
36. $7\frac{1}{2}a - 6 - 3\frac{1}{2}a = 24 - 6a.$
37. $9n + 7 + 6n + 3 = 40 - 5n.$
38. $8x + 7 - 6x - 3 = 19 - 7x.$
39. $9x - 8 = 0.$
40. $7d + 4 - 3d = 4.$

Exercises. *Algebraic Representation*

1. One number is 4 more than another number. If n denotes the smaller number, what algebraic expression represents the sum of the two numbers?

2. One number is five times as large as another number. If n represents the smaller number, what algebraic expression represents their sum? their difference?

3. One number is 3 less than another number.

   (1) If the smaller number is n, what is their sum?
   (2) If the larger number is n, what is their sum?

4. Mary and Elizabeth played a game. Mary's score exceeded that of Elizabeth by 5. What was the combined score of the two girls? (Use s for Elizabeth's score.)

EXPLANATION. The statement $8 - 6 = 2$ may be read in several ways:

(1) The difference between 8 and 6 is 2.
(2) 8 exceeds 6 by 2.
(3) 8 is 2 larger than 6.

5. Write in short form:

   (1) The difference between a and b is c.
   (2) x exceeds y by r.
   (3) x is d larger than y.

6. A man's profit, in dollars, was p. How do you represent

   (1) a profit which is \$1000 larger?
   (2) a profit which is \$2000 less?
   (3) a profit which is five times as great?
   (4) a profit which is half as great?

7. Paul's age is a. His brother is 5 years older. Express their ages 10 years from now.

8. A dozen eggs cost c cents. What is the cost of one egg? of 5 dozen eggs?

9. The cost of a suit, in dollars, was d. What is the price of 4 such suits? of n suits?

10. The cost of 4 bushels of potatoes was $3.60. What was the price of 1 bushel? of n bushels?

11. During a certain week William earned d dollars. He spent c dollars for lunches and 50 cents for his other expenses. How many cents did he have left?

12. I bought a house having a value of n dollars. The interest rate on this sum was 6 % a year. My other expenses were $400 a year. How many dollars did the house cost me per month?

13. Remember that integers which are arranged in their natural order, such as 6, 7, 8, are called *consecutive* integers. If n represents any integer, how do you represent the next two consecutive integers?

14. Study the following table of *even* integers:

$2 = 2 \times 1$ $\qquad$ $8 = 2 \times 4$
$4 = 2 \times 2$ $\qquad$ $10 = 2 \times 5$
$6 = 2 \times 3$ $\qquad$ $12 = 2 \times 6$

Evidently, any even integer may be written in the form $2 \times (\quad)$, or $2n$. Explain.

15. If $2n$ represents any even integer, what expression represents the next consecutive *even* integer?

16. Study the following table of *odd* integers:

$7 = 6 + 1$ $\qquad$ $21 = 20 + 1$
$9 = 8 + 1$ $\qquad$ $25 = 24 + 1$

Evidently, any odd integer may be represented by the expression $2n + 1$. Explain.

17. If $2n + 1$ represents any odd integer, write an expression for the next odd integer.

51. Summary. In this chapter you have learned

(1) How to solve equations by inspection.

(2) How to solve equations by rule.

(3) How to check the solution of an equation.

(4) The use of the fundamental axioms.

(5) The meaning of important terms, such as *equality, identity, equation, root, solution, members of an equation.*

Test on Chapter IV

"The Case of A and B"

In the following contest imagine that two boys are playing a game and that A represents the score of one boy, while B represents the score of the other boy. The class may be divided into two teams. The team which scores the largest number of correct answers wins the equation contest.

1. $A + B = 20.$

If A is	1	3	7	10	12	19
then B is . . .	___	___	___	___	___	___

2. $A - B = 40.$

If B is	3	7	10	15	20	39
then A is . . .	___	___	___	___	___	___

3. $A + 2B = 30.$

If A is	2	6	8	10	14	28
then B is . . .	___	___	___	___	___	___

4. $4A - 2B = 20.$

If B is	0	2	4	8	12	40
then A is . . .	___	___	___	___	___	___

5. $\frac{4}{5}A + 4B = 60.$

If A is	0	5	10	35	25	50
then B is . . .	—	—	—	—	—	—

6. $\frac{3}{4}A + 6B = 72.$

If A is	0	8	16	40	32	48	56
then B is . . .	—	—	—	—	—	—	—

7. $\frac{3}{4}A - 3B = 9.$

If B is	1	2	7	5	8	12	20
then A is . . .	—	—	—	—	—	—	—

8. $\frac{2}{3}A - B = 21 + 2B.$

If B is	1	3	5	7	11	21
then A is . . .	—	—	—	—	—	—

CHAPTER V

PROBLEMS

52. Solving Problems by Equations. Many mathematical problems, as you have seen, can be solved very quickly by using equations. In fact, the equation is the principal tool for solving such problems. In shops, laboratories, offices, banks, and factories the use of the equation is a great timesaver. It is consequently very important that you should have considerable practice in the solution of problems by equations.

Professor Albert A. Michelson, of the University of Chicago, is one of the foremost scientists of our times. He is shown working out difficult problems in his laboratory

The problems that arise in everyday life are often very complicated or technical. Such problems require a knowledge of special facts or principles which cannot be taught in ordinary algebra classes. It will be our aim, instead, to consider only problems which do not depend on technical knowledge and which can be mastered without long-continued special study. While these simpler problems are not always very "practical," you will find that the *method* used in solving them applies also in the case of more technical problems.

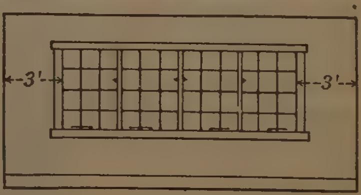
53. Simple Applied Problems. Some of the problems which follow can be solved by either arithmetic or algebra. You may work them out by arithmetic first, if you prefer. Study the illustrative examples and try to discover, in each case, in what way the equation method is better than the arithmetic solution.

Illustrative Example 1. Four casement windows of equal width are to be built in a wall 18 feet long, leaving a space of 3 feet on each side of the windows. What must be the width of each window?

Solution. 1. The missing number in this case is the *width* of each window. Hence we represent it by w.

2. We see that four of these widths, increased by 6 feet, make up the whole length of the room, which is 18 feet.

3. This relation may be written in equation form, as follows:



$$4w + 6 = 18.$$

4. Solving this equation, we have

(1) $4w + 6 = 18.$ | Given.
(2) $4w = 12.$ | S_6.
(3) $w = 3.$ | D_4.

5. We see that this solution is correct, for if we have four widths of 3 feet each, and add 6 feet, we obtain a total distance of 18 feet.

Try to summarize the five steps used in this solution.

Illustrative Example 2. John, Henry, and Fred together built a chicken house and started a small poultry business. Henry contributed as much money to the business as John, while Fred contributed as much as John and Henry together. At the end of the year the total profit of the business was found to be \$80. How much of this profit should each boy receive?

Solution. 1. Let John's profit be represented by p.

2. Then Henry's profit is also represented by p, while Fred's profit is represented by $2p$. (Why?)

3. The total profit is then represented by $4p$. Hence we have the equation $4p = 80$.

4. Solving this equation, we have

(1) $4p = 80$.	Given.
(2) $p = 20$.	D_4.

5. Since John's share is \$20, Henry's share is also \$20, while Fred's share is \$40. Also, since \$20 + \$20 + \$40 = \$80, the solution is correct.

The work for this problem may be given in condensed form, as follows:

Boy		Profit	Check	
John		p	= \$20	
Henry		p	= \$20	
Fred		$2p$	= \$40	
Solution.	1.	$4p$	$= 80$	Given.
	2.	p	$= 20$	D_4.

54. Steps Used in Solving Problems. The above illustrative examples suggest the following steps, which aid in the solution of problems stated in words:

1. Read the problem carefully. Picture the situation which it describes. Try to discover what is *required.* Represent the unknown quantity by a letter.

2. If several unknown quantities are to be found, express their *relations* in terms of a *single* letter. If convenient, make a *table* showing these relations.
3. Form the *equation* suggested by the conditions of the problem.
4. *Solve this equation* and thus find the results required.
5. *Check* your results by using the original statements of the problem.

Key Words: **missing numbers, relations, equation, solution, check.**

Note. It is very important to distinguish carefully between checking the solution of an *equation* and checking the solution of a *problem*. In the case of a problem, it is necessary to go back to the *original* statements to see whether the solution "works." For it is possible that you may have solved the equation correctly, but the equation itself may not correctly represent the conditions mentioned in the problem. You would not discover this mistake by merely checking the solution of the equation.

Exercises. *Problems*

Solve each of the following problems, using the suggestions given above.

Draw a figure whenever it is convenient. For the sake of comparison, try to solve Exercises 1 to 9 by both arithmetic and algebra.

1. A rectangular field, 100 feet wide, is to be inclosed with wire fencing. If 500 feet of fencing are at hand, how long can the field be made?

2. The length of a rectangular field is to be three times the width. Suppose that 80 rods of fence are at hand to inclose it. What must the dimensions of the field be?

3. A box is 3 feet long and 2 feet wide. How high must it be to hold 12 cubic feet?

4. Mr. Jones has on hand 100 feet of poultry netting, which he wishes to use to make a chicken yard.

(1) What would be the dimensions of the yard if he made it square?

(2) What would the dimensions of the yard have to be if the length were to be three times the width?

(3) Which arrangement would make the larger yard?

5. In my garden I wish to have a border for flowers, which is to be 3 feet wide. I have estimated that the flowers and plants which I wish to grow during the coming season will require 60 square feet of garden space. How long must the border be?

6. How high should a schoolroom be which is 30 feet long and 20 feet wide if it is to accommodate 36 pupils, 200 cubic feet of air space being considered necessary for each pupil in a schoolroom?

7. Henry and his sister were together given 90 cents each day to pay their school expenses. They had to pay 16 cents each for carfare and with the remainder had to buy their lunch. How much can each spend for lunch?

8. The graduating class of the Franklin High School wish to buy a class memorial, which is to cost $120. If there is $50 in the class treasury, how much should each of the 140 members of the class contribute?

9. In a certain city the taxi rate is 30 cents for the first mile and 20 cents for each additional mile.

(1) How far did a passenger travel whose fare amounted to $1.30?

(2) How far can a passenger ride if he wishes to spend not more than 90 cents for a taxi fare?

SUGGESTION. Let n = the number of miles traveled at the 20-cent rate.

55. Number-Relation Problems. Number problems, while of no practical importance, will serve to give you additional skill in analyzing and solving verbal problems. Study the following illustrative examples.

Illustrative Examples. Suppose that someone had written the whole numbers 1, 2, 3, 4, and so on, up to 100, and that you were asked to select from them certain numbers, provided these were described in some definite way. An indefinite number of problems could then be given, many of them of a type that could be solved readily by the equation method.

1. Select two of these numbers such that one is three times the other and their sum is 60.

Solution. Let $n =$ the first number.
Then $3n =$ the second number.
Then $4n = 60.$ (Why?)
Hence $n = 15.$
Therefore $3n = 45.$

Check. Evidently, since 45 is three times 15, and since $15 + 45 = 60$, the solution is correct.

2. Select three numbers such that the second is 2 greater than the first, and the third is equal to the sum of the first two, provided the sum of all three numbers is 36.

Solution. Let $n =$ the first number.
Then $n + 2 =$ the second number.
Then $2n + 2 =$ the third number. (Why?)
(1) $4n + 4 = 36.$ (Why?)
(2) $4n = 32.$ $S_4.$
(3) $n = 8.$ $D_4.$

Hence the required numbers are 8, 10, and 18.

Check. (Do these numbers meet all the conditions of the problem?)

Exercises. *Number Relations*

1. Three whole numbers are represented by n, $n + 2$, and $n + 3$. Their sum is 26. Find the numbers.

2. Four whole numbers are represented by n, $2n + 1$, $3n + 2$, and $4n + 6$. Their sum is 99. Find the numbers.

3. Four whole numbers are represented by n, $4n + 1$, $2n$, and $n + 9$. The sum of the first two equals the sum of the last two. Find the numbers.

4. Six times a certain number is equal to the number increased by 10. Find the number.

5. The greater of two numbers is seven times the less, and their sum is 160. Find the numbers.

6. When a certain number is multiplied by 7, and the product is decreased by 4, the result is three times the number. What is the number?

7. The sum of two numbers is 90, and the greater is eight times the smaller. Find the numbers.

8. Two numbers differ by 56, and the greater is eight times the smaller. Find the numbers.

9. Two whole numbers are such that one is 10 greater than the other and their sum is 40. Find the numbers.

10. The sum of two numbers is 28. The greater exceeds the smaller by 2. Find the numbers.

11. The sum of four numbers is 279. The first two numbers are equal. The third is twice the sum of the first two. The fourth is three times the second. What are the numbers?

12. Find two consecutive integers whose sum is 59.

SUGGESTION. The first integer is n. The second is $n + 1$.

13. Can the sum of two consecutive integers ever be an even number? Explain.

14. Verify the relations shown in the following table:

NUMBER OF CONSECUTIVE INTEGERS	ALGEBRAIC FORM	SUM
1	n	
2	$n + 1$	$2n + 1$
3	$n + 2$	$3n + 3$
4	$n + 3$	$4n + 6$
5	$n + 4$	$5n + 10$
6	$n + 5$	$6n + 15$

15. Find three consecutive integers whose sum is 63.

16. Find five consecutive integers whose sum is 110.

17. Find two consecutive even integers whose sum is 62.

SUGGESTION. Let $2n$ represent the first integer.

18. Find three consecutive even integers whose sum is 66.

19. Make a table showing the sum of two, three, four, five, consecutive even integers.

20. Find two consecutive odd integers whose sum is 44.

SUGGESTION. Let $2n + 1$ represent the first integer. Then $2n + 3$ represents the second integer.

21. The sum of three consecutive odd integers is 129. Find them.

22. Which of the following numbers might represent

(*a*) the sum of three consecutive integers?
(*b*) the sum of three consecutive even integers?
(*c*) the sum of three consecutive odd integers?

(1) 36. (2) 60. (3) 75. (4) 249. (5) 126.

56. Geometric Problems. Many problems concerning perimeters, areas, and volumes can be solved more easily by using the equation method. The following exercises will serve to increase your skill in understanding and solving such problems.

Exercises. *Geometric Problems*

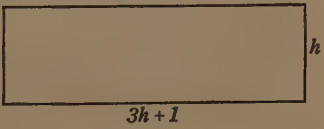
1. The perimeter of a square was found to be 100 feet. How long was one side? Solve by arithmetic and by algebra.

2. The perimeter of an equilateral triangle was equal to that of a square of side 3. Find the side of the triangle. Solve by arithmetic and by algebra.

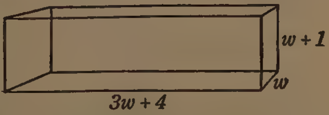
3. The sides of a triangle are represented by consecutive integers. Write a formula for the perimeter.

4. In Exercise 3, if the perimeter is 63, how long is each side?

5. A rectangle whose dimensions have the relations shown in the figure is such that if each dimension is increased by 5 the perimeter becomes 62. Find the dimensions of the rectangle.



6. It is required to make a wire model of a rectangular solid, the dimensions having the relations shown in the figure. If the total length of the wire available for this purpose is 40 inches, what must be the dimensions of the model?



7. An automobile driveway is 54 feet long and 8 feet wide. It is to be covered with cinders. If three loads of

cinders are available for the purpose, how deep a layer of cinders is possible? (1 load = 2 cubic yards.)

8. The base of a cylindrical can has an area of 14 square inches. How high must it be to have a volume of 70 cubic inches?

9. A group of boys decide to lay out a circular running track which is to be a quarter of a mile in length. What must be the radius of the track? ($C = 2\pi r$.)

10. The floor area of a rectangular store was 3200 square feet. If the store had a frontage of 120 feet, what was its depth?

11. The side of a square is $2s + 5$. If the perimeter is 100, find s and $2s + 5$.

57. Business Problems. Many business problems can be solved very easily by the use of equations. They are concerned with questions of cost, of profit and loss, of interest, discount, commission, and the like. Since percentage is of very common occurrence in such problems, we shall consider first the solution of equations having decimal coefficients.

Illustrative Example. Find a number of which 15 is 3 per cent.

Solution. Let $n =$ the required number.

Then $3\%\, n = 15.$

This may be written $.03\, n = 15,$

or $\frac{3}{100} n = 15.$

Dividing both sides by $\frac{3}{100}$, we have

$$n = 500.$$

Check. $\frac{3}{100} \times 500 = 15,$

or $15 = 15.$

Exercises. *Business Problems*

1. $5\%\, n = 20.$
2. $6\%\, n = 24.$
3. $4\%\, x = 16.$
4. $4\%\, m = 60.$
5. $10\%\, x = 90.$
6. $12\%\, x = 84.$
7. $2\%\, n = 4.$
8. $3\%\, n = 9.$
9. $2\frac{1}{2}\%\, n = 10.$

SUGGESTION. Remember that $2\frac{1}{2}\%\, n = \frac{2\frac{1}{2}}{100} n = \frac{5}{200} n.$

10. $3\frac{1}{2}\%\, x = 21.$
11. $3\frac{1}{3}\%\, n = 10.$
12. $4\frac{1}{4}\%\, c = 17.$
13. $2\frac{1}{8}\%\, x = 34.$
14. $3\frac{3}{4}\%\, x = 30.$
15. $x + .04\, x = 208.$

Solution. Exercise 15 may be written $1.04\, x = 208,$

or $\frac{104}{100}\, x = 208.$

Dividing by $\frac{104}{100}$, we have $x = 200.$

16. $x + .03\, x = 309.$
17. $n + .06\, n = 106.$
18. $c + 4\%\, c = 312.$
19. $x + 15\%\, x = 805.$
20. $p + 12\%\, p = 336.$
21. $x - .04\, x = 96.$

SUGGESTION. Exercise 21 may be written $.96\, x = 96.$

22. $n - 10\%\, n = 270.$
23. $x - 2\%\, x = 196.$
24. $c - 8\%\, c = 368.$
25. $x - .06\, x = 188.$

26. In what time will a sum of $4000 yield an income of $800, at 4% a year?

SUGGESTION. Use the formula $i = prt.$

Hence $4000 \times .04 \times t = 800.$

27. In what time will a sum of $2500 yield an income of $375, at 5% a year?

28. At what rate must $4000 be invested in order to yield a profit of $200 a year?

SUGGESTION. $\frac{r}{100} \times 4000 = 200.$

29. At what rate must \$3500 be invested in order to yield \$210 a year?

30. Using the formula $i = prt$, find, by the equation method, the missing values in the following table:

	PRINCIPAL	RATE	TIME	INTEREST
(1)	\$500	4%	2 years	
(2)	\$600	4%		\$72
(3)	\$940		2 years	\$94
(4)		6%	$1\frac{1}{2}$ years	\$97.20

Using the percentage formula $p = br$, find the missing values in the following exercises:

31. $b = 200$, $r = 5\%$.
32. $b = 88$, $r = 10\%$.
33. $b = 916$, $r = 12\%$.
34. $p = 18$, $r = 5\%$.
35. $p = 40$, $r = 2\%$.
36. $p = .84$, $r = 6\%$.
37. $p = 8$, $b = 40$.
38. $b = 2.50$, $p = 5$.
39. $b = 9.16$, $p = 36.64$.
40. $b = 8140$, $r = 2\frac{1}{2}\%$.

41. In a class of 40 pupils 9 failed. What per cent of the class failed?

42. On a certain farm the gross receipts during a certain year were \$10,000, while the total expenses were \$6270. What per cent of the receipts were the expenses?

43. A man bought a house for \$5500 and sold it for \$6200. What was his per cent of gain?

44. A farmer took 940 lb. of milk to a creamery. It was found that the milk produced 38.5 lb. of butter fat. What was the per cent of butter fat?

45. A pair of shoes sold for $4.50, after a discount of 10 % had been allowed on the marked price. What was the marked price?

46. At what price must a pair of shoes costing $6 be listed in order that after allowing a discount of 10% the firm may yet make a profit of 20 %?

Solution. Let l denote the list price.

Then $.90\,l = 1.20 \times 6.$

This may be written $\frac{90}{100}\,l = \frac{120}{100} \times 6.$

Dividing by $\frac{90}{100}$, we have $l = 8.$

That is, the list price is $8.

Check. 90% of $8 = $7.20.
120% of $6 = $7.20.

47. A man bought an automobile for $960. The following year he sold it for $500. What was his per cent of loss?

48. A house sold for $5300, after a discount of 4 % had been allowed for cash. What was the original price?

49. At what price must a contractor offer a house that cost him $5700, in order that after allowing a selling agency a commission of 5 % he may make 10 % himself?

50. An automobile agent receives a commission of 25 %. What must be the list price of the cars he sells if his firm is to make a profit of 20 %, the cost of a car being $600?

58. Everyday Problems. The problems of life are never accompanied by a rule that tells how they should be solved. Instead, one must *discover* in each case exactly what is required and what must be done to solve the problem.

The following problems, while not practical in the usual sense, are intended to give you some skill in understanding or picturing *described* situations, and in learning to apply to them the equation method.

Exercises. *Miscellaneous Problems*

1. A boy started on a trip from Buffalo to Jamestown, a distance of 72 miles. He rode eight times as far as he walked. How far did he walk?

2. Some Boy Scouts took a hike of 24 miles. They walked 3 miles farther in the morning than in the afternoon. How far did they walk in the afternoon?

3. A girl paid $5.50 for a tennis racket and three balls. The price of the racket was $4.75. Find the price of a ball.

4. The principal of a school, his wife, and his little daughter went to Washington. The total railroad fare amounted to $42.20. If the child paid half fare, what was the price of each ticket?

5. Two boys having a launch took a party on a pleasure trip for $25. The boy who owned the launch received $5 more than the other boy. How much did each receive?

6. Three grades raised $160 for the Scholarship Fund. The eighth grade raised $10 more than the seventh grade, and the ninth grade raised $20 more than the eighth grade. How much did each grade raise?

7. Three girls in the Commercial Department agreed to rent a typewriter for $12. The second girl used it three times as many hours as the first, and the third girl twice as many hours as the first. How much should each pay?

8. Three basket-ball teams played 18 games. The first two teams won an equal number of games, and the third team won 4 games. How many games did each team win?

9. A baseball nine has played 12 games and has won 3 less than twice the number lost. How many games has it won?

10. Harry has $18.75. He earned $3 of this money by washing a car and the rest of the money by mowing lawns. He received 75¢ for each lawn which he mowed. How many lawns did he mow?

11. Three boys sold 70 tickets for a football game. John sold twice as many as Bob, and Dick sold four times as many as Bob. How many did each boy sell?

12. In an athletic meet three boys represented East High School. Together they scored 22 points. The first boy scored 5 more points than the second, and the third scored as many as the other two together. How many points did each one score?

13. I bought some 5-cent stamps and three times as many 2-cent stamps. If I paid 55¢ for all, how many of each did I buy?

14. How can a bill of $3 be paid with half-dollars and quarters by using the same number of each?

SUGGESTION. Since the value of one half-dollar is 50 cents, the value of n half-dollars is $50\,n$ cents.

15. A concrete road cost $75,000. The county paid twice as much as the state, and the town paid three times as much as the state. How much did each pay?

16. Three neighbors compare their taxes. The first finds that he pays twice as much as the second, and the third finds that he pays as much as the other two together. If the sum of their taxes is $1440, how much does each pay?

17. Four boys decided to run a refreshment booth one summer. They found that the equipment would cost $85. The father of one of the boys gave them $25. How much will each boy have to pay if the expense is divided equally?

18. A man finds that it will cost him $450 to build a garage. The materials cost $290. A carpenter and his helper agree to do the work in 10 days. The carpenter is to be paid three times as much as his helper. How much does each earn per day?

19. Two boys went fishing and caught 15 fish. When they returned home, the boy coming from the larger family took 3 more than twice as many as the other one. How many did each boy take?

59. Some Old-Fashioned Problems. The problems which follow appeared in American algebra textbooks that were in use about seventy-five years ago. You will observe that some of them are very simple, while others may at first seem hard to you.

Exercises. *Old-Fashioned Problems*

1. John and Charles have 12 apples between them, and each has as many as the other. How many has each?

2. James and John have 24 peaches, and one has as many as the other. How many has each?

3. In a fruit basket there are three times as many apples as pears, and five times as many peaches as apples, — in all, 95. How many of each sort are there?

4. On a certain day a merchant paid out $2500 to three men, A, B, and C; he paid to A a certain sum, to B $500 less than the sum paid to A, and to C $900 more than he paid to A. Find the sum paid to each.

5. A boy bought an equal number of apples, oranges, and pears for 96 cents; he bought the apples at 3 cents apiece, the oranges at 4 cents, and the pears at 5 cents. How many of each kind did he buy? *Ans.* 8.

6. A man bought 3 oxen, 4 cows, and 6 calves for 260 dollars. He paid twice as much for an ox as he did for a cow, and twice as much for a cow as for a calf. How much did he give for each?
Ans. For a calf, $10; for a cow, $20; and for an ox, $40.

7. A gentleman purchased a horse, a chaise, and a harness for $230. The harness cost a certain sum, the chaise three times as much as the harness, and the horse $20 more than the chaise. Can you find the price of each?

8. A man paid a debt of $29, in three different payments; the second time he paid $3 more than he did the first time, and the third time twice as much as the second time. What was the amount of his first payment? *Ans.* $5.

9. A farmer hired a man and two boys to do a piece of work. To the man he paid $12, to one boy $6, and to the other boy $4 per week. They all worked the same length of time and received $264. How many weeks did they work?

10. One half of a post stands in the mud, one third in the water, and the remainder, which is 3 feet, is above the water. What is the whole length of the post?

(Let $6x =$ the length of the post.) *Ans.* 18 feet.

11. A farmer has three pastures for sheep. In the second pasture he has twice as many sheep as in the first, and in the third as many as in the first and second less 15, and he has in all 57 sheep. How many sheep has he in each pasture?

12. Jane has a blush rosebush, a moss rosebush, and a white rosebush, and there are in all 33 buds. The buds on the second rosebush are double those on the first, and those on the third four times those on the second. How many buds are there on each rosebush?

13. Four boys, A, B, C, and D, in counting their money, found they had together $1.98, and that B had twice as much as A, C as much as A and B together, and D as much as B and C together. How much money did each boy have?

Ans. A, 18 cents; B, 36 cents; C, 54 cents; and D, 90 cents.

14. A man deposited in a savings bank, in three deposits, a total of $80. The second deposit was double the first, and the third was equal to the first and second and $8 over. What was the sum deposited each time?

15. A man had six sons, to whom he gave $120, giving to each one $4 more than to his next younger brother. How many dollars did he give to the youngest? *Ans.* $10.

16. A person goes to an inn, where he spends 3 shillings. He then goes to a second inn and spends 9 shillings, which is three times as much as he had left. What had he at first?

17. James has twice as many dime pieces as cent pieces, and 42 cents in all. How many dime pieces has he?

18. Divide $1000 among A, B, and C so that A shall have $72 more than B, and C $100 more than A.

60. Summary. In this chapter you have learned

(1) **How to solve simple number problems.**

(2) **How to solve simple geometric problems.**

(3) **How to solve business problems.**

(4) **How to solve other applied problems.**

Supplementary Honor Work

1. Make a collection of number-puzzle problems.

2. Make a collection of problems found in old algebra textbooks.

3. Try to make a list of original problems suggested by those given in this chapter.

Test on Chapter V

In each of the following problems write the algebraic expression which represents the answer:

1. James and Henry together have s dollars. James has $16. How many dollars has Henry?

2. In how many hours can a boy walk d miles at the rate of r miles an hour?

3. Represent each of three consecutive numbers, and also their sum, if the first number is represented by n.

4. On a certain test Jane's and Harry's standings differed by $d\%$. If Jane received the lower rating, which was 75%, what was Harry's standing?

Under each of the following problems three equations are given. Select the equation which correctly represents the conditions of the given problem.

5. In a class of 32 pupils the number of girls was 6 less than the number of boys. How many boys and how many girls were there in the class?

(1) $n - 6 = 32$. (2) $2n - 6 = 32$. (3) $n + 6 = 32$.

6. A certain rectangular room required 72 feet of molding. The room was twice as long as it was wide. Find its dimensions.

(1) $2w = 72$. (2) $4w = 72$. (3) $6w = 72$.

7. John has $3.00 in the school bank at the present time, and he saves 25 cents a week. In how many weeks will he have $10.00?

(1) $25w + 3 = 10$. (2) $25w + 300 = 1000$.
(3) $25w + 3.00 = 10.00$.

8. The side of an equilateral triangle is 3 inches longer than the side of a square. The two perimeters are equal. Find the side of each figure.

(1) $3s + 9 = 4s$. (2) $4s + 3 = 3s$. (3) $3s = 4s + 12$.

Write the equation which represents the conditions of each of the following problems:

9. One man has three times as much money as another. Together they have $5400. How much money has each?

10. A farm of 325 acres is to be divided between two brothers. The older is to receive 75 acres more than the younger. How many acres should each receive?

11. After a discount of 10% on the list price had been allowed, the cost of a radio set was $126. What was the list price?

Solve completely each of the following problems:

12. I have 200 feet of wire fencing with which I wish to inclose a rectangular garden plot. This plot is to be 40 feet wide. How long may I make the plot if all the wire is to be used?

13. A class of 36 pupils, having only \$10 in its treasury, desires to purchase for the school a memorial valued at \$19. How much must each pupil in the class contribute so that the desired gift can be purchased?

14. I bought some 2-cent stamps and five times as many 1-cent stamps. If I paid 70 cents for all, how many of each did I buy?

A Modern Bank. Interior of National City Bank, New York City

CHAPTER VI

GRAPHS

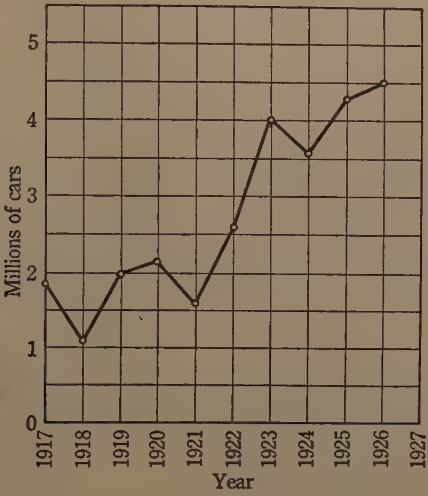
1. Statistical Graphs

61. Importance of Statistics. Modern life has become so complicated that unless a record were kept of important facts, they would soon be forgotten, and great confusion would result. Thus, in school the attendance is taken in every class from day to day. The marks of each pupil are entered in the class books of the teachers and in the general office files. A business man, in order to keep track of the progress of his business, must constantly watch the amount of sales, expenses, net profits, and the like. Income-tax returns must be based on carefully kept accounts. The Census Bureau records the growth of the population and many other items. A large number of clerks are necessary in many offices, both private and public, to take care of the nation's bookkeeping.

For these reasons it is sometimes said that we are living in a statistical age, and it is not surprising that the importance of accurate statistics is being realized more clearly from year to year.

62. Why Graphs are Used. The statistical material which appears in newspapers and magazines is usually presented in *tabular* form. The *table* is undoubtedly the most convenient form of arranging statistical facts.

Nevertheless, when a statistical table is long, or contains large numbers, it is not always easy to understand the story which the table contains. It has therefore become customary to "picture" such a table by making a drawing that represents the numbers found in the table. A drawing of this kind is called a *graph*.

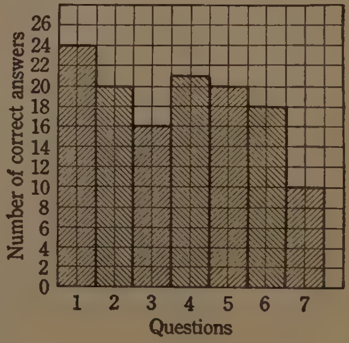


Annual production of cars and trucks

1917	**1,869,000**	**1922**	**2,659,000**
1918	**1,154,000**	**1923**	**4,087,000**
1919	**1,974,000**	**1924**	**3,618,000**
1920	**2,205,000**	**1925**	**4,337,000**
1921	**1,662,000**	**1926**	**4,480,000**

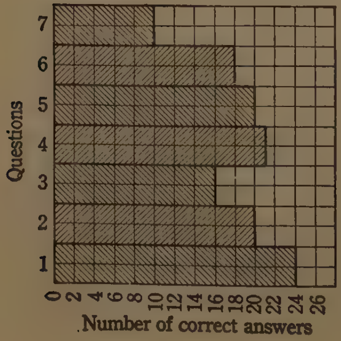
Study the graph shown above and the table which accompanies it. Which tells the story of the automobile industry more quickly?

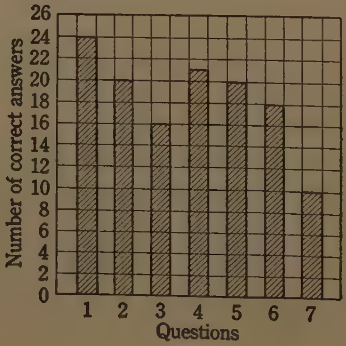
63. Kinds of Graphs. You may have observed that many other kinds of graphs are used in business and everyday life. Each graph tells a story and is based on some definite facts. The form which is chosen in any particular case depends on the facts to be pictured and on the purpose of the graph. Thus, if a graph is to be used for *advertising*, it is necessary to keep in mind the distance at which it is to be clearly seen. A graph appearing in a newspaper is not necessarily effective when used on an advertising card in a street car. Begin to make a collection of graphs which you may find in newspapers, magazines, and advertising folders. Try to understand in each case the exact meaning of the graph and the method employed in making it. A few of the most important kinds of graphs will be considered in the following pages.



64. Bar Graphs. The diagram shows one form of *bar* graph. It is based on the following "story." A group of 25 pupils took a test of seven questions. The results were to be pictured graphically so that all could see at a glance how many pupils had answered each question correctly. Can you read, or interpret, the graph?

1. How many pupils answered the fifth question correctly?





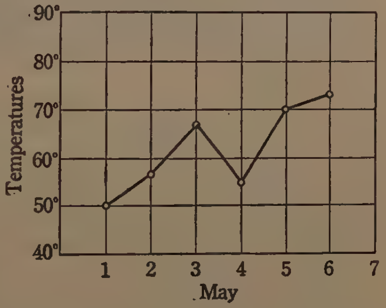
2. Which question caused the greatest difficulty?

3. Which question received the greatest number of correct answers?

In a bar diagram the bars may be arranged horizontally or vertically. They may be placed in a compact form or they may be separated. Which of the three forms shown on the preceding page would be the most effective one when seen at a distance?

65. Broken-Line Graphs. During the first six days of May, in a certain city, the average daily temperatures were as follows:

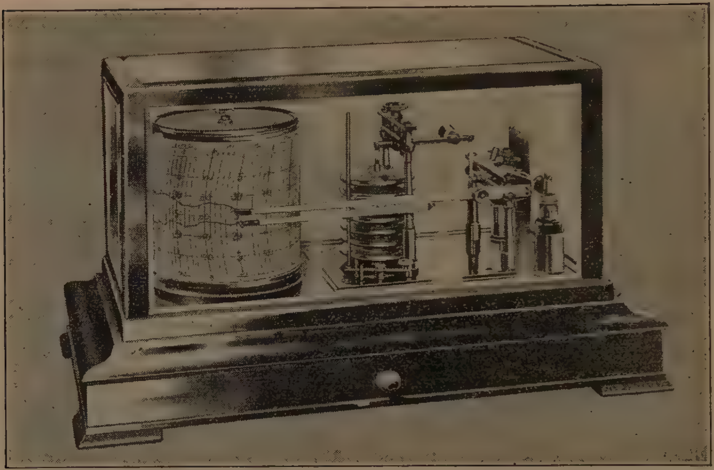
May 1	50°
May 2	57°
May 3	67°
May 4	56°
May 5	70°
May 6	73°



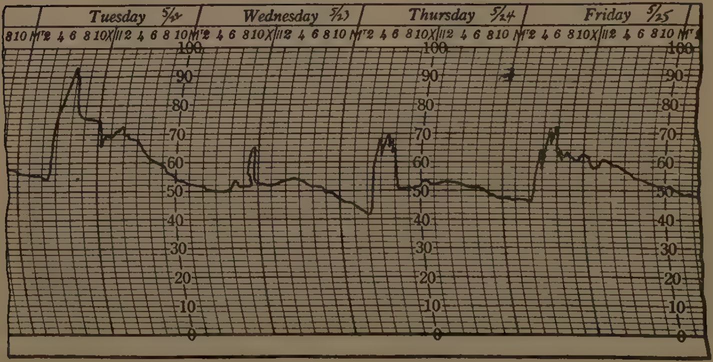
The adjoining graph pictures this table.

The line on which the days are marked off is called the *horizontal axis*, while the *vertical axis* shows the temperatures. The average temperature for each day is indicated by a point. By connecting these points in succession, as shown, we obtain a *broken-line* graph.

66. Curved-Line Graphs. When the facts to be pictured are very numerous, or when they fluctuate very much, it is often preferable to use a curved line in making the graph. Such a curve may be drawn free-hand or by means of special instruments. Thus, the temperature prevailing out of doors may change considerably from hour to hour. A complete record of these changing temperatures is made



The instrument shown in the picture makes a continuous graphic record of the temperature and of the barometric pressure



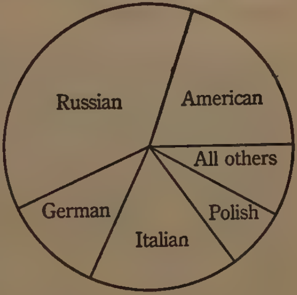
A temperature graph made by a thermograph

by a *thermograph*, which traces graphs, such as those shown above, on a special kind of graph paper.

One advantage of a *continuous* temperature record presented by a curved-line graph is that one may obtain from it the temperature that prevailed at any time during the period which is considered. In a broken-line graph one cannot be equally certain about values that lie *between* those which were used in constructing the graph.

67. The Circle Graph. When it is desired to show the relation of each item of a table to the *sum* of the items, the graphic representation which is commonly used is the *circle* graph. The adjoining diagram illustrates a common form of the *circle* graph.

It is based on the following story. In a certain school the pupils were requested to give the nationality of their parents. It was found that 19.8 per cent of the pupils were of American parentage, that 16.9 per cent had Italian parents, and so on.



Observe that the circle graph shows these facts at a glance, giving clearly the *relative* importance of each item. It is for this reason that circle graphs are frequently used in business to show each item of the total cost of production, the relative wages paid to different classes of workmen, the effect of various types of advertising, and the like.

68. Interpretation of Graphs. By means of a graph it is possible to present numerical facts quickly and in a striking way. That is why graphs are used so extensively for commercial purposes. But just as we should always use common sense in reading a newspaper article or a book that may be of interest to us, so it is necessary to guard constantly against misleading or incorrect impressions in reading a graph. This means that it is necessary, while reading a graph, to ask questions such as the following:

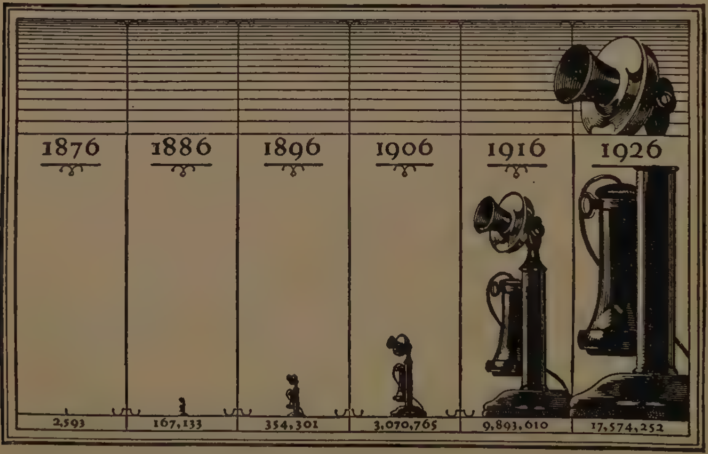
1. Does the story of the graph seem reasonable?

2. Is the graph constructed correctly?

3. Which parts of the graph have a message of particular interest?

Exercises. *Reading and Interpreting Graphs*

Examine each of the following graphs. Select any three of them for special study. Try to learn something about the story which each is supposed to tell, and then answer the accompanying questions.



Growth of the telephone business

1. The largest American telephone company expects eventually to "enable anyone anywhere at any time to pick up a telephone and talk to anyone else, quickly and at a reasonable cost."

(1) How does the graphic picture on the preceding page show the increase in the number of telephones used in this country from 1876 to 1926?

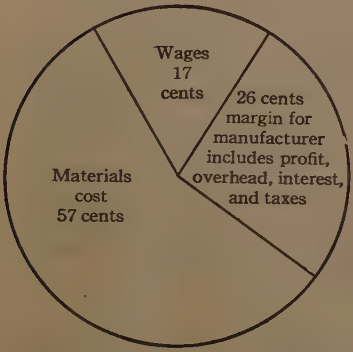
(2) Is a graphic picture of this kind as accurate as a bar diagram?

(3) Mention one advantage and one disadvantage of a pictorial graph. Why are such graphs used especially in advertising?

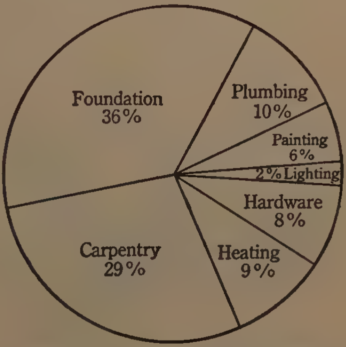
(4) What was the growth in the number of subscribers during each ten-year period indicated in this graphic picture?

(5) If the rate of increase of the last ten-year period continues until 1936, how many telephones will then be in use?

2. Study the two adjoining circle graphs.



Manufacturer's dollar. Average division of industrial receipts. Computed from data of Census Bureau

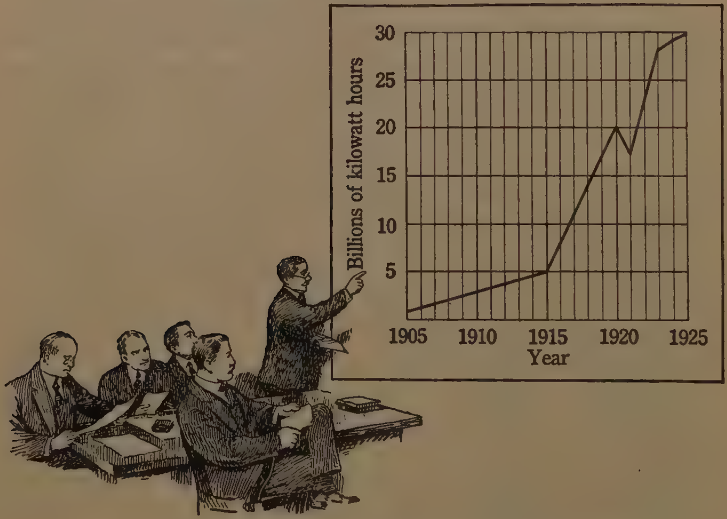


Building costs. Principal items of expense in construction

(1) Out of every dollar received by the manufacturer, according to the circle graph shown above, how many cents were expended for materials? How many for wages? How many cents remained for the payment of overhead expenses, taxes, and the like?

(2) Suppose that the total expense of building a new home is estimated at $10,000. According to the circle graph shown on page 137, what part of this sum must be set aside for the foundation? What part for carpentry? What part for plumbing?

3. Business men constantly study graphic charts such as the one shown below. Such charts are often called the "barometers of business." Why?



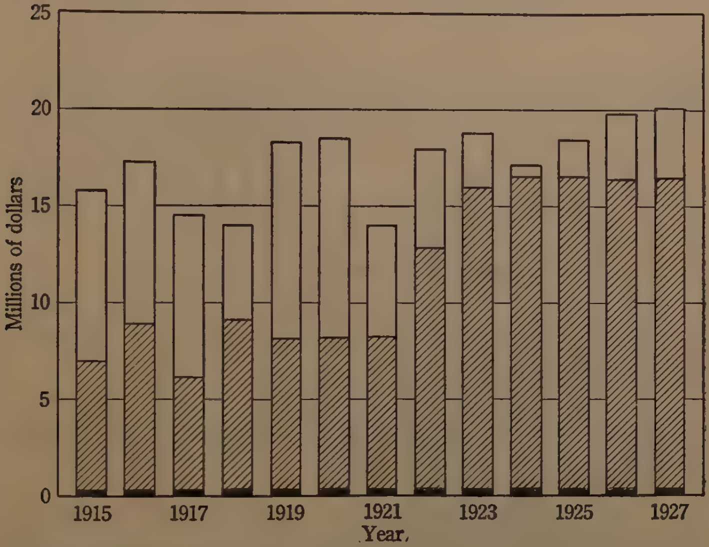
Electricity consumed for industrial power

(1) Statistics show that since 1905 the consumption of electricity for industrial power has increased to about thirty times the original amount. How is this fact shown in the graph?

(2) Can you give a reason for this remarkable increase?

(3) Which five-year period showed the greatest increase in the industrial use of electric power?

4. The bar diagram below shows the annual earnings of a large corporation since 1915, and the portion of this income which was paid out in the form of dividends.



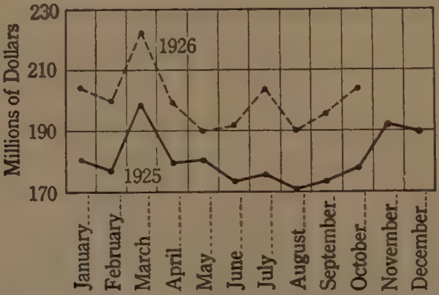
Annual earnings of a large corporation. In this chart the full height of each bar represents the total earnings in that year. The black, or lowest, section shows the amount paid out for preferred dividends, while the shaded section shows the amount of common dividends

(1) Which year showed the most marked increase in the payment of dividends?

(2) What is the outlook for continued prosperity in the case of this corporation?

5. The number of checks cashed from week to week in any community is a good "barometer of business." Compare the two broken-line graphs on the chart shown on the next page.

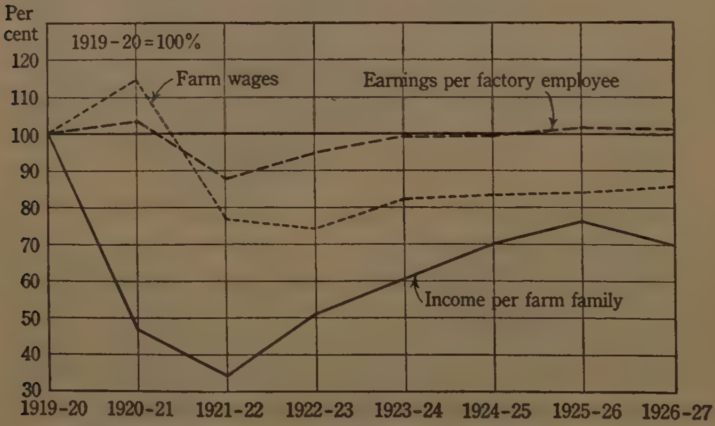
(1) How did the business of 1926 compare with that of 1925?



Checks cashed in Los Angeles. Weekly average

(2) Which month seems to indicate a heavy volume of business in Los Angeles?

6. Study the comparative graphs shown on the following chart:



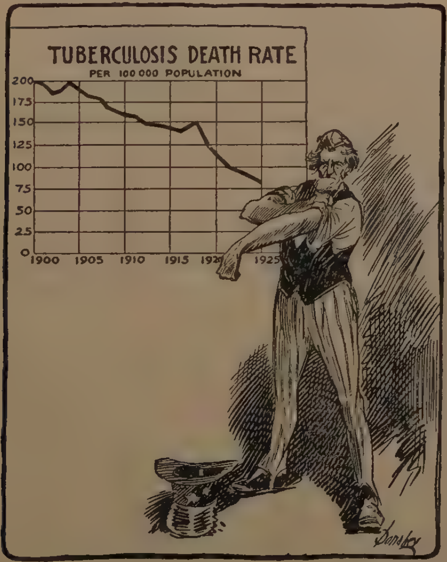
Net income per farm family,* farm wages, and earnings per factory employee

(1) What is true of the earnings of factory workers since 1923?

(2) What has been true of farm wages since 1920?

(3) In what year did the income per farm family begin to improve?

7. Graphic posters are often used in health campaigns and for effective work in advertising. Study the graphic chart shown below.



"Let us keep this black line going down!"

Courtesy of the National Tuberculosis Association

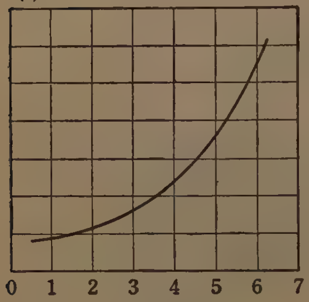
(1) What is the principal reason for the appearance of this poster in newspapers and magazines?

(2) How much did the death rate due to tuberculosis decline between 1900 and 1925?

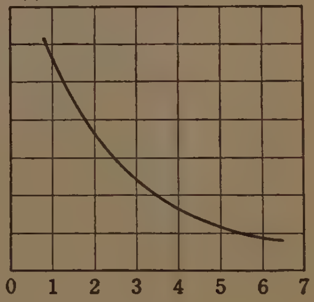
8. Examine each of the following typical graphs. Discuss the possible meaning of each if it is supposed to picture statistics relating to

(1) Population
(2) School attendance
(3) Income
(4) Expenses
(5) Cost of living
(6) Volume of business
(7) Health items
(8) Taxes
(9) Crops
(10) Exports

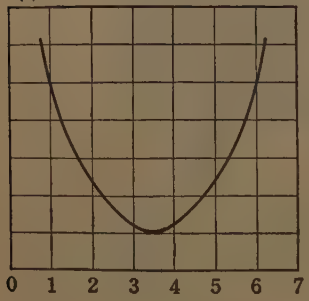
(1)



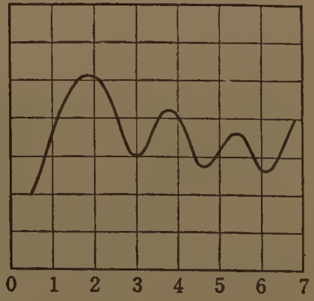
(2)



(3)



(4)



Thus, in the case of the cost of living, a graph such as (1) shows that prices are *going up*, while a graph of type (2) shows that prices are *going down*. A graph such as (3), however, means that prices decline and then rise again, while type (4) pictures *fluctuating* prices.

69. How to Make a Graph on Squared Paper. Graphs can usually be drawn most quickly and accurately by using *squared paper*. For this purpose it is possible to obtain squared paper ruled in inches or in centimeters. When the statistics to be pictured involve decimals, it is desirable to use decimally divided "inch paper" or "centimeter paper."

In most cases the graph is used to picture the facts given in a table, as was shown on page 131. In making a graph you will therefore find it helpful to keep in mind the following

Rules for Making Graphs

1. On a sheet of squared paper draw the axes. The *horizontal axis* is usually drawn at the bottom, and the *vertical axis* is drawn at the left side of the paper.
2. Using a convenient scale, mark off one set of numbers given in the table on the horizontal axis, and the other set on the vertical axis.
3. The divisions on each axis should be chosen in such a way that the numbers given in the table can be easily located.
4. It is usually best to start with zero, the numbers extending from zero to the right, and from zero upward, on the two axes.
5. In the case of a bar diagram, especially if it is to be used for advertising purposes, it is customary to draw the bars some distance apart. The space between the bars is usually made the same as the width of the bars.
6. The "story of the graph" should be explained by a suitable title written at the top or at the bottom of the graph sheet.

Exercises. *Making Statistical Graphs*

In the following exercises use any form of graphic representation which is found most convenient:

1. One of the stations of the United States Weather Bureau reported the following hourly temperature readings for a day in July, from 6 A.M. to 6 P.M.:

Time	6	7	8	9	10	11	12	1	2	3	4	5	6
Readings . .	67°	69°	71°	77°	79°	82°	85°	86°	73°	80°	83°	82°	82°

Draw a graph picturing these temperature readings. On the afternoon of that day there was a thunderstorm. How does the graph show the resulting effect on the temperature?

2. Draw a temperature graph based on the figures given in the following table:

Day in November . .	1	2	3	4	5	6	7	8	9	10	11	12
Average temperature .	66°	57°	42°	40°	38°	32°	32°	31°	36°	43°	62°	50°

3. The average monthly temperatures reported during a recent year, from January to December, in a large Eastern city, were as follows: 27°, 24°, 28°, 40°, 54°, 62°, 70°, 70°, 61°, 49°, 40°, 25°.

Represent these figures graphically.

4. The average monthly rainfall in inches, from January to December, in the same city, was reported as follows: 2.9, 2.6, 2.9, 2.5, 2.8, 3.1, 3.0, 2.9, 2.5, 2.8, 2.5, 2.7.

Draw a graph based on these figures. (Let 10 divisions of the vertical axis represent 1 inch of rainfall.)

√ 5. The average heights of American boys at various ages are shown in the following table:

Age in years . .	6	7	8	9	10	11	12	13	14	15	16
Height in inches	43.8	45.7	47.8	49.7	51.7	53.3	55.1	57.2	59.9	62.3	65.0

Draw a graph based on this table.

6. Represent graphically the statistical facts given in the following table, which gives the average heights of girls at various ages:

Age in years .	6	7	8	9	10	11	12	13	14	15	16
Height in inches .	43.4	45.5	47.6	49.4	51.3	53.4	55.9	58.2	59.9	61.1	61.6

√ 7. On a single sheet of squared paper, and using the same axes, draw the graphs of Exercises 5 and 6.

(1) During which years are the heights of boys and girls practically alike?

(2) During which years does the height of boys exceed that of girls?

8. The following table shows the average weights of American boys and girls at various heights:

Height in inches	48	50	54	56	58	60	62	66	68	70
Weight of boys (in pounds) .	53	58	71	78	85	94	104	129	139	147
Weight of girls (in pounds) .	52	58	71	79	89	101	114	129	138	144

(1) Using the same set of axes, draw two comparative graphs, one representing the weights of boys, and the other the weights of girls.

(2) From your graphs determine the approximate weight of a boy 59 inches in height, and that of a girl 55 inches in height.

✓ 9. Eight algebra classes took a test on equations. Two weeks later they took the same test a second time. The table shows the average percentage of correctness obtained in these two trials.

Class	1	2	3	4	5	6	7	8
First trial	92	91	88	85	84	83	80	74
Second trial	97	94	90	86	84	82	78	70

(1) On a single sheet of squared paper, and using the same axes, draw comparative graphs which are based on these figures.

(2) How many of the classes improved their average on the second trial?

\ 10. The following table shows the amount required in various years to purchase what one dollar bought in 1890:

1890	$1.00	1910	$1.23
1895	.93	1915	1.38
1900	1.02	1920	2.75
1905	1.11	1925	2.31

Draw a graph based on this table.

✓ 11. The average retail price of certain foods is shown in the following table. The figures represent cents per pound.

Year	1918	1919	1920	1921	1922	1923	1924	1925	1926
Butter	58	68	70	52	48	55	52	55	53
Sugar	10	11	20	8	7	10	9	7	7
Round steak	37	39	40	34	32	34	34	35	36

Using the same axes, draw a set of graphs showing the fluctuations of these prices.

12. How the cost of building, per square foot of floor area, has increased is shown in the following table:

1919	$4.62	1922	$5.78	1925	$6.47
1920	6.34	1923	5.92	1926	7.27
1921	6.06	1924	6.33	1927	7.16

Draw a graph representing these facts.

13. The average monthly consumption of gasoline, in millions of barrels, for ten successive years, beginning with 1917, was as follows: 4.6, 6.2, 6.8, 8.4, 9.0, 10.6, 13.0, 15.4, 18.6, 21.2. Represent these statistics by a graph.

14. The amounts of money, in billions of dollars, deposited in savings banks in certain years since 1914 are shown in the following table:

1914	4.9	1922	7.2
1916	5.1	1924	8.4
1918	5.5	1926	9.6
1920	6.5	1928	10.2

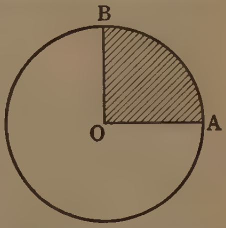
Make a graph based on this table.

15. The number of acres of farm land in the United States at ten-year intervals from 1860 to 1920, expressed in millions, was as follows: 294, 408, 536, 623, 839, 879, 956. Represent these statistics by a graph.

SUGGESTION. When a statistical table contains large numbers, it is evidently impossible to picture these graphically without using a convenient *scale*. Thus, to picture 294,000,000 acres, we may let one division of the vertical axis represent 100,000,000 acres. Then 294,000,000 is represented by 2.94, or approximately 2.9, of these divisions. Using this scale, the other numbers given above may be replaced by 4.1, 5.4, 6.2, 8.4, 8.8, and 9.6.

16. The average daily number of messages, in thousands, handled by one large telephone company in six successive years, beginning with 1922, was as follows: 264, 286, 304, 355, 371, 387. Represent these statistics by a graph.

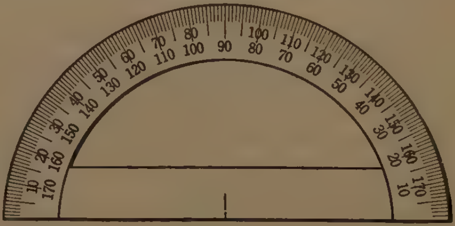
70. How to Make a Circle Graph. In making a circle graph, we must divide the circle into parts which represent the given items, by drawing lines from the center of the circle. These parts are sometimes called *sectors* of the circle.



Thus, in the figure the radii OA and OB form a *sector* which represents one fourth of the circle.

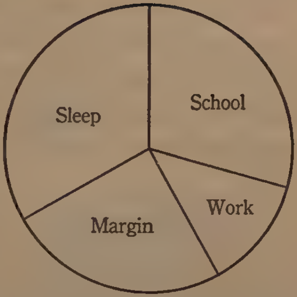
A quick way of dividing a circle into the required parts is based on the use of a special instrument called the *protractor*. A form of the protractor is shown in the figure below.

If two protractors of this kind are placed so as to form a circle, the circumference of this circle will show 360 equal divisions, or *degrees*. This is written 360°. By connecting each of these divisions with the center of the circle, we obtain 360 small sectors. It is easy, therefore, to divide the circle into sectors that correspond to any desired fraction or decimal of 360°. By using 120° we obtain one third of the circle. Similarly, 90° is one fourth of the circle, 108° is 30 per cent of the circle, and so on.



Illustrative Example. The pupils of a health class were asked to prepare a "time budget" showing how they spent their time on a certain day. Each pupil was requested to find out how large was the "margin of time" left over when allowance was made for sleep, school hours, and other items.

William Smith reported that he spent his day as follows: sleep, 8 hours; school, 7 hours; outside work, 3 hours; margin, 6 hours.



In making a circle graph based on this time budget, he arranged his work as follows to obtain the sector necessary for each item:

	NUMBER OF HOURS	FRACTIONS OF DAY	DEGREES IN REQUIRED SECTOR
Sleep	8	$\frac{1}{3}$	120°
School	7	$\frac{7}{24}$	105°
Work	3	$\frac{1}{8}$	45°
Margin	6	$\frac{1}{4}$	90°
Total	24	$\frac{24}{24} = 1$	360°

Exercises. *Making Circle Graphs*

1. In a class of 36 pupils the following marks were given at the end of a month: A, 4 pupils; B, 9 pupils; C, 12 pupils; D, 8 pupils; and E, 3 pupils. Show by a circle graph how the marks of the class were distributed.

2. A family budget of $2400 was divided as follows: rent, $600; food, $700; clothing, $300; operating expense, $300; savings, $100; miscellaneous, $400. Show by a circle graph how the income of this family provided for the items listed above.

3. A study of the ages of pupils in a certain school showed that 70% were of normal age, while 20% were over-age pupils and 10% were under-age pupils. Draw a circle graph picturing this distribution.

√ 4. The total expenditures of a large power company during a recent year were divided as follows: labor, 21%; fuel, 19%; material, 14%; taxes, 11%; depreciation, 9%; interest, 8%; dividends, 12%; surplus, 6%. Represent these items by a circle graph.

? 5. The "tax dollar" of a large city during a recent year was divided as follows: public instruction, 40¢; health, police, and fire departments, 20¢; public works, 17¢; debt service, 11¢; administration, 5¢; parks, 4¢; miscellaneous, 3¢. Show by a circle graph how the taxes of this city were used.

6. The cost of operating the average passenger automobile during a recent year was found by experts to be about 10 cents a mile. This cost was divided as follows: gas and oil, 18%; depreciation, 31%; maintenance, 12%; tires, 10%; license, 6%; garage, 8%; interest and insurance, 15%. Draw a circle graph picturing these items.

7. During a recent year the world production of rayon silk was nearly 220 million pounds. This production was distributed as follows: United States, 29%; Italy, 16%; Germany, 12%; England, 11%; France, 8%; other countries, 24%. Draw a circle graph picturing this distribution.

8. The total receipts from the Federal income tax, as reported by the United States Treasury Department during a recent year, were derived as follows: individual taxpayers, 44%; manufacturing corporations, 27%; public utilities, 9%; trade corporations, 8%; banking corporations, 7%; other corporations, 5%. Draw a circle graph based on these figures.

9. In 1927 the United States exported to foreign countries merchandise valued at \$1,616,000,000. This foreign trade was divided as follows: Europe, 47%; North America, 24%; Asia, 13%; South America, 10%; Australia, 4%; Africa, 2%. Draw a circle graph based on these figures.

10. In a recent year the total steel production of the principal manufacturing countries was distributed as follows: United States, 50%; Germany, 15%; England, 9%; France, 8%; all other countries, 18%. Draw a circle graph showing these facts.

2. Graphs of Formulas

71. Picturing a Formula. It was shown in a preceding chapter that it is often possible to obtain a formula from a table. By reversing the process, it is possible to change a formula back into a table by substituting in the formula certain numerical values for the letters appearing in the formula. From such a table we may readily obtain a picture of the formula by using the graphic method explained previously. The following simple illustrations will show how the graphic method is applied in the case of formulas.

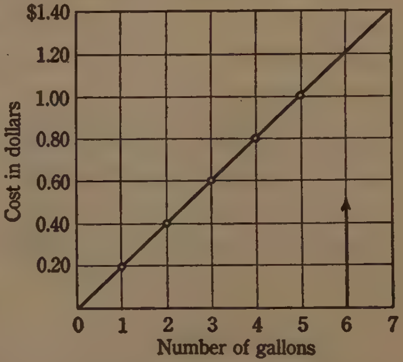
Illustrative Example 1. *The Cost Graph.* Suppose that gasoline costs 20¢ a gallon. Then the cost in dollars (c) of any number of gallons (g) is given by the formula

$$c = 0.20\,g.$$

By substituting in this formula successive values of g, we may make a table of corresponding values, as follows:

If $g =$	0	1	2	3	4	5	⋯
then $c =$	0	0.20	0.40	0.60	0.80	1.00	⋯

These corresponding sets of numbers can be pictured on squared paper, as here shown. On the horizontal line we record the number of gallons, and on the vertical line we indicate the cost. Thus, above 1 we locate a point opposite 0.20; above 2 we locate a point opposite 0.40; and so on. By connecting these points we obtain a straight-line graph.



Such a graph can be used as a "ready reckoner." That is, the cost of any number of gallons can be read directly from the graph if the graph is extended sufficiently far. Thus, to find the cost of 6 gallons, we locate the number 6 on the horizontal axis. The vertical line drawn from that point meets the graph at a point which is opposite $1.20. This is the required answer.

Similarly, if we wish to find how many gallons can be bought for a given sum of money, we locate first the given sum of money on the vertical axis. The horizontal line from that point meets the graph at a point which is directly above the desired number on the horizontal axis.

In other words, this cost graph does all the computing, or "reckoning," which would ordinarily have to be done by actual multiplication. Much time may therefore be saved by using the graph of a formula.

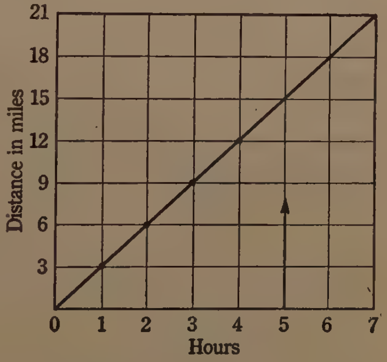
Illustrative Example 2. *The Uniform-Motion Graph.* Suppose that a man walks away from his home at the uniform rate of 3 miles an hour. Then, after 2 hours of walking, he will be 6 miles from his home; after 3 hours, 9 miles; and so on. Evidently, the distance which he covers in any number of hours (h) may be obtained from the formula $d = 3h$.

The table below shows a number of corresponding values of h and d:

If $h =$	0	1	2	3	4	5	...
then $d =$	0	3	6	9	12	–	...

Using the plan explained above, we obtain the following *motion graph.* Observe that one space on the vertical axis represents 3 miles.

Hence, to find the distance covered in 5 hours, locate 5 on the horizontal axis. On the vertical line drawn from that point locate the point on the graph which will enable you to find the corresponding distance. It is situated opposite 15. Similarly, if the distance is known, we can find from the graph the corresponding number of hours.

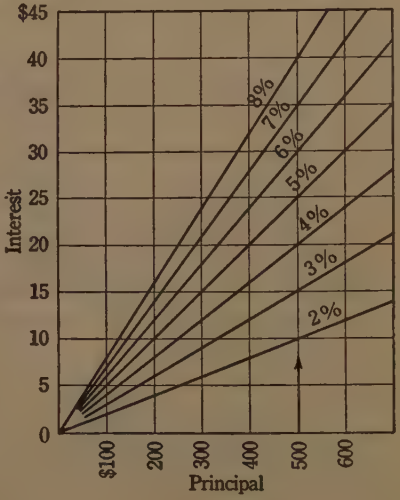


Illustrative Example 3. *The Graph for Simple Interest.* In a previous chapter you studied the interest formula $i = prt$. If a sum of money is put at interest for *one* year, at the rate of 4 per cent, the formula becomes

$$i = .04\,p.$$

Thus, if $p = \$100$, $i = \$4$; if $p = \$500$, $i = \$20$; and so on.

In the same way we might make a formula for any other rate of interest. For 5 per cent we have $i = .05\,p$; for 7 per cent, $i = .07\,p$; and so on. By using the method explained above, we may then construct a set of graphs for any desired values of r, as suggested by the adjoining diagram.



If these graphs were extended sufficiently far, we could use them to find the interest on any convenient sum of money, at the rates represented in the diagram.

Thus, the interest on \$500 at 2 per cent is \$10; at 3 per cent, \$15; at 4 per cent, \$20; at 8 per cent, \$40.

Exercises. *Graphs of Formulas*

1. Using the method explained above, draw a graph of the formula $d = 10\,h$.

2. If the cost of a certain article is 8 cents a pound, the cost of any number of pounds is represented by the formula $c = 8\,n$. Draw a graph of this formula.

3. Draw a graph of the interest formula $i = .05\,p$, if p represents the principal and i the interest, the rate being 5 per cent.

Find from the graph (1) the value of i when $p = \$250$: (2) the value of p when $i = \$30$.

4. If the price of gasoline is 21 cents a gallon, show by a graph the cost of any number of gallons up to 10 gallons.

(1) From the graph determine the price of 6 gallons; of 9 gallons.

(2) How many gallons can be bought for $1.47?

5. The formula for the perimeter of a square is $p = 4\,s$, where s represents the length of one side. Draw a graph of the formula.

From the graph find (1) the value of p when $s = 5\frac{1}{2}$; (2) the value of s when $p = 30$.

6. Draw a graph of the circumference formula $C = \pi d$.

SUGGESTION. Represent C on the vertical axis and d on the horizontal axis. Let $\pi = 3\frac{1}{7}$, and find the values of C for different values of d.

7. A man earns $7.50 a day. Draw a graph showing his wages after 1 day, 2 days, 3 days, and so on.

8. A pupil's savings from week to week, in cents, were expressed by the formula $s = 40 + 6\,w$. Explain. Draw a graph of the formula. Determine by the graph how long it will take to save one dollar.

9. Construct a graph enabling you to read off the number of feet in a given number of yards. (Use the formula $f = 3\,y$.)

10. Make a graph by which you may find the number of centimeters in a given number of inches. (One inch = 2.5 centimeters, nearly.)

11. Show graphically the relation between pounds and kilograms. (One kilogram = 2.2 pounds, nearly.)

NOTE. Observe that when we draw the graph of a mathematical formula, we are really picturing in each case the relationship between changing quantities, which the formula expresses.

3. THE GRAPHIC SOLUTION OF PROBLEMS

72. Solving Problems Graphically. So far you have solved numerical problems either by ordinary arithmetic or by means of a table or by using the equation. There are problems, however, which can be solved more conveniently by using the *graphic* method. A few simple illustrations will show how graphs are used in solving problems and how this graphic method compares with the other methods.

Illustrative Example 1. Two neighbors, A and B, start on a trip at the same time and travel in the same direction. A travels at the rate of 20 miles an hour, while B's rate is 25 miles an hour. After how many hours will they be 15 miles apart?

Solving by Arithmetic. Since B covers 5 miles more than A each hour, it is evident that the two men will be 15 miles apart after 3 hours.

Solving by Table. By making a table showing the progress of A and B from hour to hour, we readily see that it will take 3 hours before the two men are 15 miles apart.

Hours	1	2	3	4	5	6
Distance of A . . .	20	40	60	80	100	—
Distance of B . . .	25	50	75	100	125	—

Solving by the Equation Method. Representing the required number of hours by h, we see that

$25h =$ the distance covered by B,

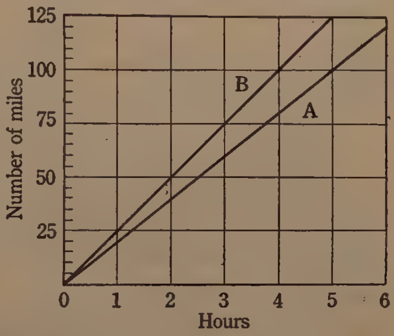
while $20h =$ the distance covered by A.

Hence we have $25h - 20h = 15.$

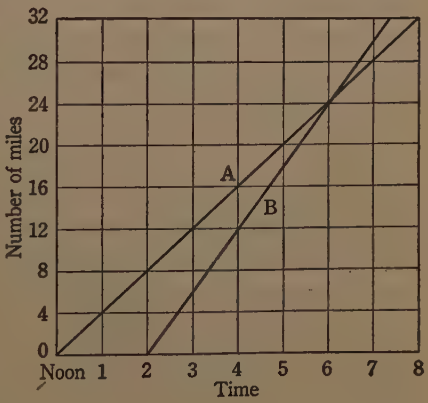
That is, $5h = 15,$

and $h = 3.$

Solving by the Graphic Method. The two straight lines in the diagram show the distances covered by A and B, respectively, from hour to hour. An inspection of the figure shows that after 3 hours the two men are 15 miles apart.



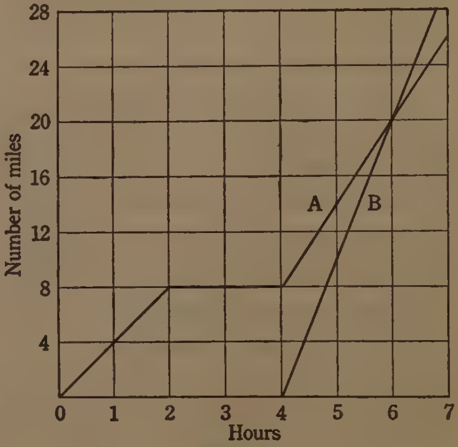
Observe that by either arithmetic or the equation method we obtain a definite answer to a *particular* question, while the table and the graph enable us to solve a *group of related problems.* The graphic method is more *general,* however, than the tabular method, since we can read at once from a graph the answers to many questions not suggested by the table. Thus, in the example above, how far apart will the men be after $4\frac{1}{2}$ hours?



Illustrative Example 2. A starts from his home at noon and walks at the rate of 4 miles an hour. B starts from the same place at 2 P.M., going in the same direction at the rate of 6 miles an hour. After how many hours will B overtake A, and how far from the starting point?

The diagram shows that B overtakes A in 4 hours, and 24 miles from the starting point.

Illustrative Example 3. A and B start at the same place, going in the same direction. A travels at the rate of 4 miles an hour, for 2 hours. Then he rests for 2 hours. He continues his trip at the rate of 6 miles an hour. B starts 4 hours after A and travels at the rate of 10 miles an hour. After how many hours will he overtake A, and how far from the starting point?



Can you interpret the graphs? What is the meaning of the horizontal portion of A's graph? What answers are suggested by the graphs?

Exercises. *Solving Problems by Graphs*

1. A grocer has two kinds of oranges which sell at 40 cents and 50 cents a dozen respectively. Make a cost graph for each kind, using the same axes for both graphs. From these graphs find out how many more oranges of the first kind can be bought for $4 than of the second kind.

2. Henry saves 50 cents a week, while John saves 75 cents a week. Draw graphs representing the savings of each boy from week to week. From these graphs find out how soon John will have saved $2 more than Henry.

3. Mary saved $2 a week. Jane was given $8 on her birthday and then she was able to save $1 each week.

Draw graphs showing how much each girl had from week to week. How soon will the two girls have the same amount of money?

4. James saved 80 cents a week. His brother William started 4 weeks later, saving at the rate of $1.20 each week. Find out by means of graphs after how many weeks the savings of the two boys will be the same.

5. Mr. Smith bought a new automobile for $2400. If its value decreased at the rate of $400 a year, show by a graph the value of the automobile from year to year, for 5 years.

6. A storage tank contained 1000 gallons of gasoline. If 50 gallons are used each day, show by a graph the number of gallons remaining from day to day.

7. Two men, A and B, travel along the same road, in the same direction. A walks 4 miles an hour and B rides at the rate of 10 miles an hour. They start from the same place, but A starts 3 hours earlier than B. Show graphically how soon B will overtake A.

8. Two cities, A and B, are 350 miles apart. A train leaves A and travels toward B at the rate of 40 miles an hour. At the same time a train leaves B and travels toward A at the rate of 30 miles an hour. Determine graphically after how many hours the two trains pass each other.

SUGGESTION. On the vertical axis mark off a distance representing 350 miles. Call the extremities of this distance A and B. Then one train starts at A and the other at B.

9. A and B start from the same place and travel in the same direction. A travels at the rate of 10 miles an hour. B starts 2 hours later. At what rate must B travel in order to overtake A in 4 hours?

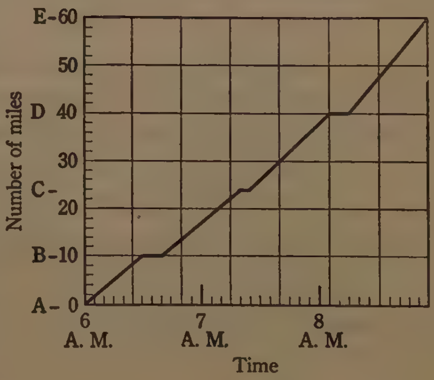
10. Graphs may be used to picture a railroad time-table, as shown in the diagram below.

Study the graph and tell what the horizontal lines mean. If a second train left station *A* at 7 o'clock, its graph could be drawn by using the same axes. Similarly, if a train left station *E* and traveled toward *A*, its graph could be drawn by laying off the miles in the opposite direction.

MILES	STATION	TIME SCHEDULE
0	*A*	Lv. 6 A.M.
10	*B*	Arr. 6.30 A.M. Lv. 6.40 A.M.
24	*C*	Arr. 7.20 A.M. Lv. 7.25 A.M.
40	*D*	Arr. 8.05 A.M. Lv. 8.15 A.M.
60	*E*	Arr. 8.55 A.M.

11. Secure a railroad time-table and show graphically the progress of all passenger trains in each direction between two stations, from 6 A.M. to 6 P.M.

12. Two men, A and B, start at the same time and travel along a certain road in the same direction toward the same town. A starts 24 miles away and walks at the rate of 4 miles an hour. B starts only 21 miles away and travels at the rate of 3 miles an hour. Show graphically the progress of the two men, and determine when one overtakes the other.



13. An algebra pupil found that it took him 15 minutes to study certain rules and 2 minutes to do one of the exercises following the rules. At that rate, show graphically

how much time he would require to complete the lesson for any number of exercises up to 15.

14. A printer finds that it costs him $3 to set up the type for a school "program" and that he can then print the program at the rate of 100 copies for 50 cents. He decides to charge $1 for a hundred copies. Draw a graph showing the actual cost of any number of copies *to the printer*. Also, using the same axes, draw a graph of the selling price, for any number of copies. How many hundred copies must be ordered in order that the printer may "break even"?

15. (*For recreation.*) Suppose that a ship leaves New York for Liverpool each day at noon; also, that a ship leaves Liverpool for New York each day at noon. If it takes exactly 7 days to cross the ocean in either direction, make a graphic chart showing the progress of these ships for a period of 2 weeks. In that case, how many ships coming from Liverpool will pass a particular ship which started from New York?

73. Summary. In this chapter you have learned

(1) How to make and interpret statistical graphs, such as bar graphs, broken-line graphs, curved-line graphs, and circle graphs.

(2) That graphs have many important uses, such as the following:

- (*a*) They tell a statistical story at a glance.
- (*b*) They enable us to make comparisons.
- (*c*) They enable us to tell in advance the probable trend of events.
- (*d*) They often call attention to conditions that can be improved or corrected.

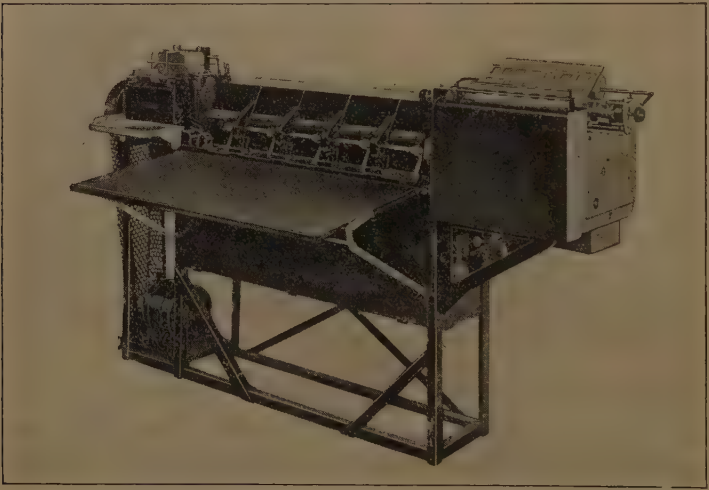
(3) How to make and use the graph of a formula.

(4) How to solve problems graphically.

Supplementary Honor Work

1. Make a collection of graphs found in newspapers and periodicals. Display these on a large sheet of cardboard and bring them to the classroom.

2. Make a graphic poster representing facts of interest to your school. You may use items relating to attendance, enrollment, games, school grades in different subjects, honor roll, bank deposits, health campaigns, and the like.



A modern tabulating machine which quickly sorts out thousands of statistical items with the aid of specially punched cards

3. Try to find out how a tabulating machine is used in sorting out quickly a large number of statistical items.

4. Read an account explaining how statistics are collected and recorded by the United States Census Bureau.

5. Make a study of the scales which are used on graphic charts to picture very large numbers.

References

ELDERTON, W. P. and E. M. Primer of Statistics. A. and C. Black, London, 1910.

KING, W. I. The Elements of Statistical Method. The Macmillan Company, New York, 1915.

Test on Chapter VI

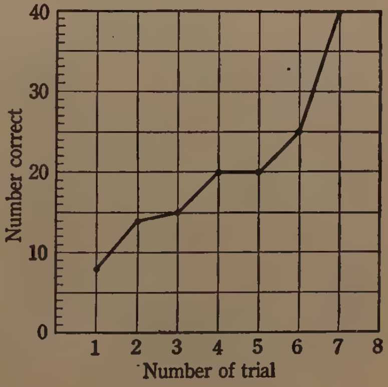
1. An algebra class was given a group of 40 practice exercises which were to be completed in 6 minutes. The accompanying graph shows a pupil's progress in 7 attempts to make a perfect score.

(1) What was his highest score?

(2) What was his lowest score?

(3) On which trial did he show no improvement?

(4) Which trial resulted in the greatest improvement of his score?



2. If butter costs 50 cents a pound, make a graph showing the cost of any number of pounds.

3. The normal weights of American boys and girls at various ages are as follows:

Ages in years	9	10	11	12	13	14	15	16	17	18
Weight of boys (in pounds) . .	59	65	70	77	85	95	107	121	125	128
Weight of girls (in pounds) .	57	62	69	78	89	98	106	112	113	114

Draw a graph showing the weights of *either* boys *or* girls.

4. Draw a graph of the formula $p = 4w + 5$, using in succession the following values of w: 0, 1, 2, 3, 4, 5, 6.

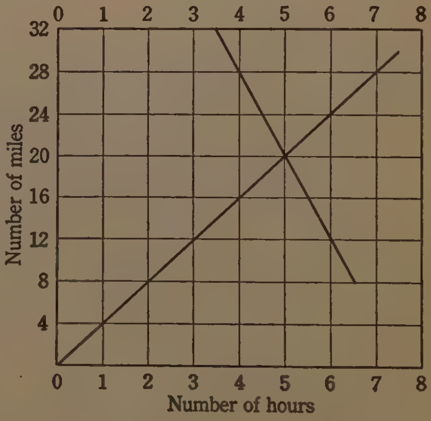
5. Represent graphically the related values given in the following table:

If r is . .	0	1	3	4	7	9	10
then d is . . .	1	4	10	13	22	28	31

(1) From the graph find the value of d when $r = 5$.

(2) From the graph find the value of r when $d = 19$.

6. The accompanying graph tells the following story: Two boys start at different times from two places which are 32 miles apart, and travel toward each other. One boy walks at the rate of 4 miles an hour. The other boy starts $3\frac{1}{2}$ hours later, riding a bicycle at the rate of 8 miles an hour.



Study the graph and answer the following questions:

(1) How many hours after the second boy starts will the two boys meet?

(2) When they meet, how many miles will each boy have traveled?

CHAPTER VII

SIGNED NUMBERS

1. EVERYDAY USES OF SIGNED NUMBERS

74. Representing Numerical Changes. An algebra class took a test on equations. The same test was given again a week later, in order that the improvement of the class in solving equations might be measured. The records of six of the pupils are shown in the following table:

PUPIL	FIRST SCORE	SECOND SCORE	COMPARISON OF SCORES
Adams, John	95	97	+ 2
Brown, Mary	94	95	+ 1
Carroll, Henry	92	90	− 2
Davis, Jane	90	87	− 3
Edwards, Grace	86	89	+ 3
Farnum, William . . .	80	75	− 5

You will observe that John Adams did *better* on the second test by 2 points. This is indicated by writing + 2 in the third column of the table. Similarly, Mary Brown made a gain of 1 point. Hence her improvement is represented by + 1. Henry Carroll, however, "went down" on the second test. He *lost* 2 points. This is indicated by writing − 2 in the third column. A similar interpretation may be given for the remaining cases in the table. Try to explain why an improvement, or a gain, is represented by a *plus* sign, while a loss is indicated by a *minus* sign.

75. Signed Numbers. You will now be able to understand the following definitions.

Positive number. A number preceded by a plus sign is called a positive number.

Thus, + 4, + 7, and + 10 are positive numbers.

Negative number. A number preceded by a minus sign is called a negative number.

Thus, − 5, − 4, and − 7 are negative numbers.

Positive and negative numbers are often called *signed numbers*. Since they are used very commonly in algebra, they are also called *algebraic numbers*.

76. Measuring the "Ups and Downs" of Everyday Life. The world in which we live is a world of *changes*. Many values continually *fluctuate*. Like the waves of the sea they rise and fall. This is true, for example, of the hourly or daily temperatures in the course of a year. It is also true of wholesale and retail prices, of exports and imports, of business profits and losses, of census figures, crop statistics, taxes, stock-exchange quotations, attendance figures, and innumerable other items. In all these cases it is often found convenient to indicate the nature of the *fluctuation* or change by a corresponding signed number. Positive and negative numbers are used quite commonly for that purpose. The following exercises are intended to give you practice in recording changes by means of signed numbers.

Exercises. *Using Signed Numbers*

1. The table at the top of the next page shows the record of Richard and Howard in five tests. How does Richard's record compare with Howard's in each of these tests?

Test	I	II	III	IV	V
Howard	60	80	65	70	90
Richard	70	60	75	70	86
Comparison . . .	+ 10				

2. The number of baseball games won by each of six teams in two successive years is shown in the following table. What improvement did each team show?

Team	I	II	III	IV	V	VI
First year	40	30	20	15	10	14
Second year	42	26	25	14	10	17
Improvement	+ 2	− 4	+ 5	− 1	0	+ 3

3. During a railroad strike the retail prices of certain commodities, in cents, changed as follows:

	BUTTER	EGGS	ORANGES	LETTUCE	POTATOES	SIRLOIN
Old price . .	65	60	75	10	100	40
New price . .	80	70	100	15	150	45
Change . . .	+ 15	+ 10	+ 25	+ 5	+ 50	+ 5

In each case indicate by a signed number the exact nature of the change in price.

4. The number of men employed by a big corporation during the first six months of two successive years was as follows:

Month . . .	JANUARY	FEBRUARY	MARCH	APRIL	MAY	JUNE
First year . .	4320	5672	6423	6240	5420	4500
Second year .	4400	5700	6549	6032	5060	4027
Comparison. .	+ 80	+ 28	+ 126	+ 208	+ 360	+ 473

Show by signed numbers how the employment figures of the two years compare.

5. Study the following table, which is based on a report of the New York Stock Exchange:

STOCKS	LAST QUOTATION	NET CHANGE
Adams Express	297	-1
Anaconda Copper	$68\frac{1}{4}$	$+2$
Atlantic Refining	124	-1
Bethlehem Steel	$63\frac{1}{2}$	$+1$
Equitable Trust Co.	500	$+9$
Illinois Central pf.	137	-2
New York Central	$136\frac{3}{8}$	$+\frac{3}{8}$
Radio Corporation of America	$182\frac{1}{2}$	$+13\frac{1}{2}$
Southern Pacific	$121\frac{1}{2}$	$+\frac{1}{2}$

In the last column are found numbers preceded by the sign $-$ or $+$. The meaning is obvious. For example, in this case "$+2$" means that certain shares gained, or went up, \$2 per share, and "-1" means that certain other shares lost, or went down, \$1 per share since the publication of the previous report. How many of the stocks listed above show a gain?

77. Absolute Value. From the illustrations given above it may be seen that a signed number, such as $+4$, may be interpreted as meaning "a *gain* of 4," while -4 means "a *loss* of 4." In either case the number of units gained or lost is the same, namely 4.

This is indicated by saying that $+4$ and -4 have the same *absolute value*. The *absolute value of a signed number*, or its *numerical value*, is its value regardless of its sign. Observe that to each positive number there corresponds a negative number of the same absolute value.

The ordinary numbers of arithmetic are, of course, always regarded as positive. Thus, 6 has the same meaning as $+6$. Moreover, it is customary to read signed numbers as they would be read in arithmetic. Thus, $+6$ is usually read "plus 6," and -6 is read "minus 6."

78. The Relation of Positive and Negative Numbers. If a man earns \$40 a week and then pays out the same sum to meet his expenses, it is clear that he has nothing left of his weekly earnings. Let us try to represent that fact by signed numbers.

Money which we earn is *added* to our resources, while money which we pay out is *subtracted* from our resources. Hence earnings are represented by *positive* numbers, while expenses are indicated by *negative* numbers. In the above illustration an income represented by + \$40 is offset by expenses represented by − \$40. The *combined* effect of having an income of \$40 and expenses of \$40 is zero. This may be written in brief form as follows:

$$(+40) + (-40) = 0.$$

It is easy to see that this relation applies to *any* positive number and its *corresponding* negative number. That is,

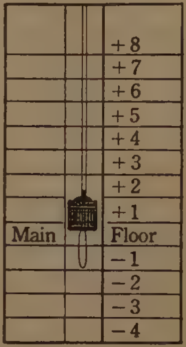
$$(+n) + (-n) = 0.$$

Hence we have the important fact that a *positive number plus the corresponding negative number equals zero.*

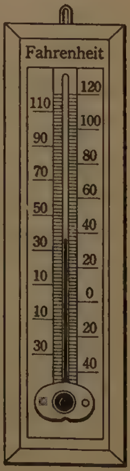
For this reason, positive and negative numbers having the same absolute value are said to be *opposites.* They destroy, or cancel, or neutralize each other. This is the reason why signed numbers may be used in many situations involving opposites.

79. Picturing Signed Numbers. The fact that signed numbers are used to measure "ups and downs" and to designate opposites also suggests a variety of simple methods of *picturing* signed numbers. Some of these will become clear from the following illustrations:

Illustrative Example 1. Let the figure suggest an elevator shaft. The horizontal partitions indicate the successive floors of the building. Let these floors above or below the main floor be numbered 1, 2, 3, and so on, as shown. An elevator starting from the main floor may make its trips in either of two directions. If it goes up to the first floor *above*, it has reached its goal by a "rise of 1"; if it goes down to the first floor *below*, it has reached its goal by a "drop of 1." Hence we may indicate the successive floors *above* the main floor by consecutive *positive* integers, and those *below* the main floor by consecutive *negative* integers. Thus, with reference to the main floor, the fifth floor up is numbered + 5, and the second floor down is numbered − 2.

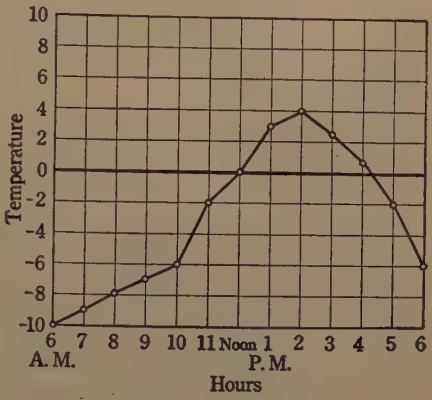


Illustrative Example 2. This is exactly the idea which underlies the number scale of a *thermometer*. The zero point on a thermometer evidently corresponds to the street level or the main floor mentioned above. Everybody knows the meaning of such expressions as "5 above zero" or "10 below zero."



Signed numbers are commonly used to designate temperatures above or below zero. Thus, − 10° means a temperature of 10 degrees below zero, while + 10° means 10 degrees above zero. In the case of temperatures above zero, the positive sign is usually omitted. With this agreement we can readily distinguish between 8° and − 8°.

This method of designating temperatures is of importance in making and interpreting temperature graphs such as the one shown in the adjoining diagram.



What was the temperature at 6 A.M.?

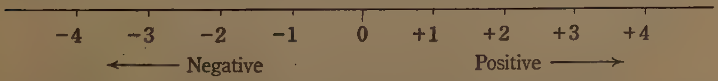
When was the temperature zero?

During which hours did the temperature rise?

During which hour did it begin to fall?

What is the highest temperature shown in the graph?

Illustrative Example 3. A number scale such as the one which is marked off on a thermometer may be in any desired position. In case of a horizontal line, it is customary to regard as positive the distances which extend to the *right* of the zero point, and as negative the distances which extend to the *left* of that point, as is shown in the following diagram. Such a diagram is called *a scale of signed numbers.*



Evidently it is constructed by marking off equal divisions, on a line of unlimited length, both to the right and to the left of some starting point, and numbering the divisions as shown. Fractions, either positive or negative, would be represented by points located between the integral division points.

A number scale may be used to explain not only the fundamental rules of signed numbers but also important applications based on these numbers. Thus, distances extending to the *east* or *north* are regarded as positive, while distances extending to the *west* or *south* are considered negative.

Since signed numbers are used to designate opposite *directions*, they are often called *directed numbers*.

Exercises. *Applications of Signed Numbers*

1. An elevator starting at the ground floor goes up to the tenth floor and then descends 6 floors. Where is it then? Draw a diagram showing its motions and express its final position in terms of signed numbers.

SUGGESTION. $(+10)+(-6)=+4$.

2. An elevator starts from the ground floor. It goes up 20 floors and then down 15 floors, then up 3 floors and then down 10 floors. Draw a diagram showing its motions and express its final position in terms of signed numbers.

3. A motor boat can make 25 miles an hour in still water. If it must make headway against a current that has a velocity of 10 miles an hour, how fast is the boat advancing? Explain.

4. An airplane, when flying at the rate of 100 miles an hour, encounters a head wind that has a velocity of 40 miles an hour. At what rate is the airplane advancing? Explain.

5. On a certain day Mr. Smith was notified by his bank that his checking account showed deposits of \$240, but withdrawals of \$260. What was his "balance"?

6. What is meant by a balance of $-\$10$?

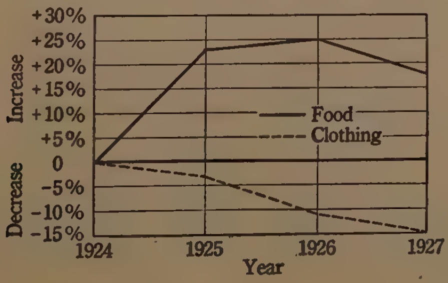
In each of the following exercises draw a graph showing the changes in temperature for the indicated period of time:

7. 7 A.M., − 6°; 8 A.M., − 3°; 9 A.M., − 2°; 10 A.M., 0°; 11 A.M., 2°; noon, 7°; 1 P.M., 8°; 2 P.M., 8°; 3 P.M., 7°; 4 P.M., 5°; 5 P.M., 2°; 6 P.M., 1°.

8. 6 A.M., 8°; 8 A.M., 9°; 10 A.M., 12°; noon, 15°; 2 P.M., 16°; 4 P.M., 8°; 6 P.M., 0°; 8 P.M., − 4°; 10 P.M., − 7°; midnight, − 10°.

9. Noon, 10°; 2 P.M., 9°; 4 P.M., 6°; 6 P.M., 5°; 8 P.M., 3°; 10 P.M., 2°; midnight, 0°; 2 A.M., − 2°; 4 A.M., − 5°; 6 A.M., − 10°; 8 A.M., − 7°; 10 A.M., 1°; noon, 4°.

10. Signed numbers are often used to show changes with reference to some value assumed as a standard of comparison. This is illustrated by the graph given below, which shows changes in the food and clothing prices from 1924 to 1927. The *zero line* represents the prices of June 1, 1924. Observe that two years later food prices had risen to a point 25% above the prices of June 1, 1924. Can you interpret the graph? What happened to clothing prices during the period considered?



Changes in food and clothing prices

11. The average attendance at a certain school during one week was 490. What was the actual attendance from day to day if the fluctuation of the attendance, *with reference to that average,* was represented by these numbers: $+3$, -4, $+5$, $+7$, -21? Draw a graph showing these changes. Let the *zero line* represent the average attendance.

✓12. In a certain city the average tax rate per \$1000, during a period of 20 years, was found to be \$20.90. During the following five years the rate changed as follows: \$20.80, \$21.10, \$22.40, \$21.00, \$20.85. Express the fluctuation of the rate with reference to the average rate, and make a graphic picture of these changes.

2. Adding and Subtracting Signed Numbers

80. Addition. A few simple examples will serve to explain the addition of signed numbers.

1. If a man first gains \$5, and then also gains \$3, his total gain is \$8.

That is, $(+5)+(+3)=+8.$

2. If a man first loses \$5 and then loses \$3, his entire loss is \$8.

That is, $(-5)+(-3)=-8.$

3. If a man gains \$5 and then loses \$3, there is still a resulting gain of \$2.

That is, $(+5)+(-3)=+2.$

4. If a man first loses \$5 and then gains \$3, there is a resulting loss of \$2.

That is, $(-5)+(+3)=-2.$

Hence we see that in the addition of two signed numbers these four cases may arise:

(1) $(+5)+(+3)=+8$.
(2) $(-5)+(-3)=-8$.
(3) $(+5)+(-3)=+2$.
(4) $(-5)+(+3)=-2$.

81. Rules for Addition. From the four illustrations given above we infer the following rules:

1. *To add two signed numbers having like signs, find the sum of their absolute values and prefix to this sum their common sign.*
2. *To add two signed numbers having unlike signs, find the difference between their absolute values and prefix to it the sign of the number which has the greater absolute value.*

NOTE. The sum of two signed numbers is often called their *algebraic sum*.

Exercises. *Addition*

Add the numbers in each of the following cases:

1.	2.	3.	4.	5.	6.
$+10$	-20	-40	-100	$+7\frac{1}{2}$	-10.14
-5	$+15$	-50	$+80$	$-5\frac{1}{2}$	$+8.25$

7. Complete the following statements:

(1) The sum of two *positive* numbers is always a ------ number.

(2) The sum of two *negative* numbers is always a ------ number.

(3) The sum of a positive and a negative number is *either* positive *or* negative, according as the ------ number or the ------ number has the greater absolute value.

8. Add $+5$ to each of the numbers in the following line:

$$+4,\ +6,\ +7,\ +10,\ +20,\ +47,\ +19.$$

To each of the numbers in the line above add -10.

9. Add -6 to each of the numbers in the following line:

$$-7,\ -2,\ -1,\ -12,\ -20,\ -41,\ -53.$$

To each of the numbers in the line above add $+12$.

10. A man deposits $100 in a bank and later checks out $60 against his account. Express his balance as the sum of two signed numbers.

Solution. $(+100)+(-60)=+40$. Explain.

11. If a man has $210 in the bank and through an oversight checks out $215, express his balance as the sum of signed numbers. What is meant by "overdrawing an account"?

12. The assets of a bank were $20,000,000 during a certain year, while its liabilities were $17,500,000. Express its net resources in terms of two signed numbers.

13. A firm failed in business. Its assets were $100,000, but its liabilities were $144,000. State its financial condition in terms of signed numbers.

Add the following:

14.	15.	16.	17.	18.	19.
$+2$	-6	-15	-20	$+16$	-1
$+7$	-3	$+10$	$+17$	$+7$	$+2$
-5	$+5$	-5	$+5$	-20	$+1$
$\underline{-2}$	$\underline{+1}$	$\underline{+6}$	$\underline{+6}$	$\underline{-40}$	$\underline{-2}$

EXPLANATION. To add *several* signed numbers, we usually find it convenient to add all the positive numbers, then to add all the negative numbers, and, finally, to find the sum of these two results.

Add in each of the following cases:

20. $+7, -4, +3, -2, +8, -10$.

21. $-40, 60, -80, 100, -200$.

22. Combine $(+10)+(-5)+(-4)+(-10)+(-9)$.

23. Combine $(-7)+(-2\frac{1}{2})+(+7\frac{1}{2})+(-10)$.

Find the missing number in each case:

24. $(+7)+(\quad)=-3$.

25. $(-9)+(\quad)=+10$.

26. $(+10)+(\quad)=0$.

27. $(\quad)+(-2)=+6$.

28. $(\quad)+(-5)=-11$.

29. $(\quad)+(-4)=0$.

30. A man deposits $300 in a bank and later checks out $250. He then deposits $50 more and checks out $80. Express his balance as the sum of signed numbers.

31. A man has real estate worth $10,000, and bank deposits amounting to $9000. There is a mortgage of $5000 on his property, and he has other debts amounting to $4500. Express his net resources in terms of signed numbers.

Practice Exercises

The following table is to be used for practice in adding signed numbers, according to the directions given on the next page:

	I	II	III	IV	V	VI	VII
A	+8	−5	+10	−9	−20	+100	+7
B	+7	−4	−6	+7	+15	−1	−7
C	−4	+3	−2	−5	+19	+2	−1
D	−14	−15	+25	−19	−13	−5	+20
E	+1	−10	−3	+16	−4	+1	−10

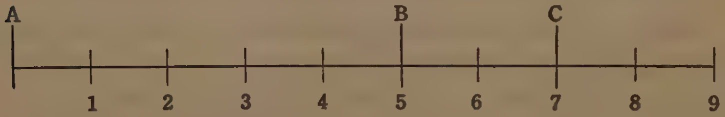
DIRECTIONS. (1) *Add the numbers of row A to the corresponding numbers of row B.*

(2) *Add the corresponding numbers of rows B and C; of rows C and D; and so on.*

(3) *Add the corresponding numbers of rows A, B, and C; of rows B, C, and D; and so on.*

(4) *Add the corresponding numbers of column I and column II; of columns II and III; and so on.*

82. Subtraction. Let the points A, B, and C, in the diagram, represent three towns on a straight road. B and C are 5 miles and 7 miles, respectively, from A. In that case



the distance separating B and C can be found by *counting backward* from C to B or by *counting forward* from B to C. The result is the same in both cases, namely, 2 miles. That is, $7 - 5 = 2$.

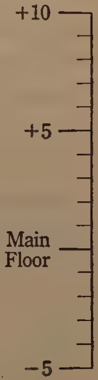
Again, suppose that you buy 81¢ worth of groceries and give the grocer \$1. He then makes change by saying "81, 85, 90, \$1," while laying down 4¢, 5¢, and 10¢. Thus he finds the value of \$1 − 81¢ by counting *forward from the subtrahend* (81¢) *to the minuend* (\$1). In this business method of subtracting we may think of the *subtrahend* (81¢) as the *starting point*, while the *minuend* (\$1) represents the *goal*.

Illustrative Example 1. How does an elevator get from the fifth floor above the ground floor to the seventh floor above the ground floor? Evidently by going up two floors.

That is, $$(+7) - (+5) = +2.$$

Illustrative Example 2. To go from the seventh floor above to the second floor below, an elevator must go down nine floors.

That is, $(-2)-(+7)=-9.$



Illustrative Example 3. If an elevator has to go from the second floor below to the seventh floor above, it must go up nine floors.

That is, $(+7)-(-2)=+9.$

Illustrative Example 4. To go from the second floor below to the fourth floor below, an elevator must go down two floors.

That is, $(-4)-(-2)=-2.$

That is, *in subtracting one signed number from another, we may find the difference by counting from the subtrahend to the minuend. If we count in the positive direction, the result is positive; but if we count in the negative direction, the result is negative.*

Oral Exercises. *Subtraction*

Perform the following subtractions by the "goal method" explained above; that is, use the subtrahend as the starting point and the minuend as the goal.

1.	3.	5.	7.	9.	11.
20	-15	-60	-41	-1	$+4\frac{1}{2}$
10	$+10$	-30	16	$+1$	$-2\frac{1}{2}$.
2.	**4.**	**6.**	**8.**	**10.**	**12,**
$+8$	-40	20	17	-18	-19.13
-4	-50	100	-2	-13	$+11.14$

13. A man compares his bank balance at the end of each month with the corresponding balance of the preceding year and enters both in two parallel columns as shown in the table. In each case find the difference between the corresponding balances.

	This Year	Last Year	Difference
(1)	$+100$	$+80$	----
(2)	-40	-20	----
(3)	$+50$	-10	----
(4)	-2	$+80$	----

Subtract in each of the following cases:

14. $(+7)-(+5)$. 16. $(-2)-(+2)$. 18. $(-8)-(-2)$.
15. $(+5)-(-2)$. 17. $(+1)-(-1)$. 19. $(-20)-(+20)$.

20. Which number, when diminished by -7, equals $+6$?

21. The noon temperatures for one week were 17°, 12°, 18°, 4°, $-2°$, 0°, 5°. Find the difference between any two consecutive noon temperatures.

22. An airplane occupied in succession the following elevations: 1000 feet, 800 feet, 1200 feet, 1600 feet. Find the difference between any two consecutive elevations.

83. Relation between Addition and Subtraction. Compare the following sets of additions and subtractions:

(1) $(+7)+(-5)=+2$.	(1) $(+7)-(+5)=+2$.
(2) $(-7)+(-5)=-12$.	(2) $(-7)-(+5)=-12$.
(3) $(+7)+(+5)=+12$.	(3) $(+7)-(-5)=+12$.
(4) $(-7)+(+5)=-2$.	(4) $(-7)-(-5)=-2$.

Hence it is clear that subtracting $+5$ gives the same result as adding -5. Similarly, subtracting -5 gives the same result as adding $+5$. These relations will become clearer by considering that taking away an income of \$5 is the same as causing a loss of \$5. Also, if someone cancels a debt of \$5 which you owe, he has really made you a present of \$5. In general,

$$-(+n) = +(-n),$$

and

$$-(-n) = +(+n).$$

That is, *subtracting a signed number is the same as adding the corresponding opposite number.*

84. Rule for Subtraction. You will now be able to understand the following rule:

> *To subtract one signed number from another, change the sign of the subtrahend and then proceed as in addition.*

This method of subtracting is called the *reversal* method, since we change, or *reverse*, the sign of the subtrahend from − to +, or from + to −. In practice we may use either the goal method or the reversal method explained above. In the latter case, the change, or *reversal*, of sign is to be performed only *mentally*.

Thus, $(+8) - (-4)$ may be changed *mentally* into $(+8) + (+4)$.

Exercises. *Subtraction*

Perform the following subtractions by the reversal method:

1. THINK:

$$\begin{array}{r} +8 \\ \underline{+4} \end{array} \qquad \begin{array}{r} +8 \\ \underline{\text{plus} \quad -4} \\ +4 \end{array}$$

2. THINK:

$$\begin{array}{r} -10 \\ \underline{+2} \end{array} \qquad \begin{array}{r} -10 \\ \underline{\text{plus} \quad -2} \\ -12 \end{array}$$

3. THINK:

$$\begin{array}{r} -5 \\ \underline{-3} \end{array} \qquad \begin{array}{r} -5 \\ \underline{\text{plus} \quad +3} \end{array}$$

4.

$$\begin{array}{r} -20 \\ \underline{+10} \end{array}$$

5.

$$\begin{array}{r} -10 \\ \underline{-10} \end{array}$$

6.

$$\begin{array}{r} -1 \\ \underline{+7} \end{array}$$

7.

$$\begin{array}{r} -12 \\ \underline{7} \end{array}$$

8.

$$\begin{array}{r} 12 \\ \underline{-5} \end{array}$$

9.

$$\begin{array}{r} -1 \\ \underline{+1} \end{array}$$

10. A man had a balance of -100 dollars at the end of a certain month. A month later his balance was $+200$ dollars. How much better off was he then?

11. On a certain day the average temperature in New York was 12°. The following day it was $-2°$. What was the difference in temperature for these two days?

Perform the following subtractions:

12. $(+20)-(-10)$.
13. $(-40)-(+20)$.
14. $(-50)-(-40)$.
15. $(-1\frac{1}{2})-(+3)$.
16. $(+7\frac{1}{2})-(-5)$.
17. $(-100)-(+50)$.
18. $(-6.5)-(+3.4)$.
19. $(+17.14)-(-13.03)$.

20. Combine $(+10)+(+8)-(-2)+(-3)-(-4)$.

21. Combine $(-40)-(-20)+(+30)-(+30)$.

Find the missing numbers in each case:

22. $-7-(\qquad)=2$.
23. $8-(\qquad)=-1$.
24. $-10-(\qquad)=16$.
25. $-12-(\qquad)=1$.
26. $40-(\qquad)=20$.
27. $4-(\qquad)=0$.
28. $6-(\qquad)=-10$.
29. $-8-(\qquad)=12$.

30. Which number, when increased by -4, equals $+10$?

31. Which number, when diminished by $+5$, equals -7?

32. The sum of two numbers is -12. One of the numbers is 20. Find the other number.

33. The sum of two numbers is 40. One of the numbers is -100. Find the other number.

34. The sum of three signed numbers is -40. Two of the numbers are 4 and 25. Find the third number.

3. MULTIPLYING AND DIVIDING SIGNED NUMBERS

85. Multiplication. Consider the following preliminary explanation. If John is 12 years old now, 2 years ago his age was $12 - 2$, and 2 years from now it will be $12 + 2$. Hence it is customary to regard as *positive* a period of time coming *after* a certain fixed date, and as *negative* a period of time coming *before* that date.

Thus, dates in history are often expressed as coming after or before the birth of Christ. Instead of 500 A.D. and 300 B.C., one might use $+500$ and -300 respectively.

Illustrative Example 1. If a man's profit in business is $4 a day, what will his profit be after 3 days? Evidently $12.

That is, $(+3) \times (+4) = +12.$

Illustrative Example 2. If a man's expenses are $4 a day, in 3 days his expenses will amount to $12.

That is, $(+3) \times (-4) = -12.$

Illustrative Example 3. If a man's wages are $4 a day, 3 days ago he had earned $12 less than he has today.

That is, $(-3) \times (+4) = -12.$

Illustrative Example 4. If a man loses $4 a day, 3 days ago he was $12 better off than today.

That is, $(-3) \times (-4) = +12.$

86. Rule for Multiplication. From these illustrations we infer the following rule:

To multiply one signed number by another:

1. *Find the product of their absolute values.*
2. *Make the product positive if the two signed numbers have like signs; make the product negative if the two signed numbers have unlike signs.*

When *several* numbers are to be multiplied together, the multiplications may be performed in the order in which the numbers are written, or we may use a more convenient order.

Example 1. $(+5) \times (-4) \times (-3) = (-20) \times (-3) = +60.$

Example 2. $(-25) \times (-6) \times (-4) = (-25) \times (-4) \times (-6)$

$$= (+100) \times (-6)$$

$$= -600.$$

Observe that if the number of negative factors is *even*, the product is positive; if *odd*, it is negative.

Exercises. *Multiplication*

Find the product in each of the following cases:

1.	2.	3.	4.	5.	6.
-6	$+10$	-7	$+40$	$+2\frac{1}{2}$	-6.1
$+4$	-4	-5	-60	-4	-2.4

7. Multiply each of the numbers in the following line by $+3$:

$$-6, \quad +7, \quad -1, \quad +10, \quad -12, \quad -20, \quad +5.$$

8. Multiply each of the numbers in the preceding line by -2; by $+7$; by -1; by -10; by $+4$.

Perform the following multiplications:

9. 4×15.

10. $3 \times (-10)$.

11. $(-5) \times 16$.

12. $(-10) \times 26$.

13. $(-19) \times (+20)$.

14. $(-1.5) \times (+18)$.

15. $17 \times (-12)$.

16. $(+2.1) \times (-1.2)$.

17. $(-6) \times (-20)$.

18. $(-71) \times (-15)$.

19. $(-1.2) \times (-10)$.

20. $(-5.6) \times (-7.3)$.

In the following exercises state the conditions of each problem by means of signed numbers:

21. A train is going east at the rate of 40 miles an hour. Where will it be 3 hours from now? Where was it 3 hours ago?

22. A train is going west at an average rate of 40 miles an hour. Where will it be 3 hours from now? Where was it 3 hours ago?

23. The temperature is rising 4° per hour. At that rate, what change in temperature will occur in 3 hours?

24. The temperature is falling 4° per hour. At that rate, what change in temperature will probably occur in 3 hours?

25. The shares of a certain stock suffered an average decline of 2 points a month during the last year. How does the present value of a share compare with its value 12 months ago?

26. The following table is based on reports of the New York Stock Exchange. The first column shows the number of shares sold on the day of the report. The last column gives the change in the value since the previous report. Compute the total gain or loss involved in the transfer of the shares listed below, no allowance being made for brokers' fees.

SALES	NAME OF STOCKS	NET CHANGE
800	Am. Locomotive	$+\frac{3}{4}$
1900	Am. Sugar Refining	$+1\frac{1}{4}$
1300	Am. Woolen	$+1\frac{1}{2}$
137,200	Bethlehem Steel	-1
5600	California Petroleum	$-\frac{1}{2}$
100	Central R. R. of N. J.	$+8$
2400	National Biscuit	$+13\frac{1}{2}$
300	Western Pacific	-2

EXPLANATION. In writing the indicated product of two signed numbers, the multiplication sign may be omitted. Thus $(+4) \times (-5)$ may also be written $(+4)(-5)$.

Find the product in each case:

27. $(-25) \times (-20) \times 4 \times 5$.

28. $(-8) \times (-2) \times (-1) \times (+3)$.

29. $(-6)(+5)(-1)(-5)$.

30. $(+20)(-8)(+\frac{1}{2})(-5)$.

31. $(-2)(-2)(-2)(-2)$.

32. $(-1)(-1)(-1)(-1)$.

33. Name two numbers which might have -12 as their product.

34. If -2 is one factor of 4, what is the other factor?

35. If -1 is one factor of -3, what is the other factor?

36. If $+10$ is one factor of -20, what is the other factor?

87. Division. Division is the *inverse* of multiplication. For example, since $4 \times 3 = 12$, it is evident that $12 \div 4 = 3$. We may therefore derive the following statements from the corresponding cases in multiplication:

(1) $\frac{+12}{+4} = +3$.

(2) $\frac{-12}{-4} = +3$.

(3) $\frac{+12}{-4} = -3$.

(4) $\frac{-12}{+4} = -3$.

Check each of these results by multiplying the divisor by the quotient.

88. Rule for Division. From the above illustrations we infer the following rule of signs for division:

> *If two algebraic numbers have like signs, their quotient is positive; if they have unlike signs, their quotient is negative.*

Exercises. *Division*

Numbers 1 to 12, oral

Perform the following divisions:

1. $(-100) \div (-4)$.
2. $(-40) \div (+10)$.
3. $(+35) \div (-7)$.
4. $(-2) \div (-1)$.
5. $(-4) \div (-4)$.
6. $(-9.1) \div (+1.3)$.
7. $18 \div (-2)$.
8. $49 \div (-7)$.
9. $(-60) \div 5$.
10. $(-100) \div 25$.
11. $(-15) \div (-5)$.
12. $(-2) \div (-2)$.
13. $(-50) \div 10$.
14. $10 \div (-50)$.
15. $40 \div (-8)$.
16. $8 \div (-40)$.
17. $12 \div (-4)$.
18. $(-4) \div 12$.
19. $(-12) \div 4$.
20. $(-12) \div (-4)$.
21. $(-60.8) \div 4$.
22. $(-18.4) \div (-.2)$.
23. $50 \div (-.02)$.
24. $(-5) \div (-5)$.

25. Which number, when divided by -7, equals 28?

26. Which number, when multiplied by -7, equals -28?

EXPLANATION. The *average* of a set of signed numbers is found exactly as in the case of the ordinary numbers of arithmetic.

Thus, the average of $+8$ and -4 is found by dividing the sum of these numbers by 2. That is, since $(+8)+(-4)=+4$, the average is $(+4) \div 2$, namely, $+2$.

27. Find the average of 10, -20, 15, -40, 100.

28. Find the average of -200, 460, -480, 1240.

29. During eight successive hours the temperature readings changed as follows from hour to hour at a certain place: 14°, 12°, 8°, 6°, 0°, $-2°$, $-4°$, $-10°$. What was the average hourly temperature during this period of time?

30. The noon temperatures, during a certain week, were 17°, 8°, 5°, 0°, – 2°, – 9°, – 2°. What was the average noon temperature?

31. Registration figures in five election districts showed the following changes when compared with the corresponding figures of the preceding year: – 42, + 12, – 32, + 57, – 40. What was the average gain or loss?

32. A number of business transactions resulted as follows: $1200 gain, $500 loss, $1000 gain, $200 gain, $800 loss. What was the average gain or loss?

EXPLANATION. When you find the average of a set of numbers, it is evident that some of the numbers must be larger than the average, while the remaining numbers must be smaller than the average. The amount by which any one of the numbers differs from the average is sometimes called its *deviation* from the average. Evidently, some of these deviations must be *positive*, while the others must be *negative*.

Thus, the average of 7, 12, 19, and 30 is 17. The deviations of these numbers from 17 are – 10, – 5, + 2, and + 13 respectively. Observe that the sum of these deviations is zero.

In general, it can easily be shown that *the sum of the deviations of a set of numbers from their average is zero.*

33. The following table shows the daily expenses of a pupil for one week, and the daily deviations from the average. Can you interpret each item?

DAY	CENTS SPENT	DEVIATIONS FROM AVERAGE	
1	13		– 31
2	27		– 17
3	45	+ 1	
4	34		– 10
5	81	+ 37	
6	17		– 27
7	91	+ 47	
Sum	308	+ 85	– 85
Average . .	$\frac{308}{7} = 44$	Check: (+ 85) + (– 85) = 0	

34. Find the average of 40, 39, 41, 38, 37, and 33. By following the method of Exercise 33, show that the sum of the deviations of these numbers from the average is zero.

35. Find the average of 50, 54, 45, 51, 45, 43, 61, and 67. Show that the sum of their deviations from the average is zero.

36. Find the average of 40, – 24, 60, 100, – 70, and – 10. Show that the sum of their deviations from the average is zero.

Practice Exercises

Try to complete these exercises within 5 minutes.

Find the sum, the difference, and also the product in each of the following cases:

1. $\begin{array}{r}+3\\ \underline{+5}\end{array}$

2. $\begin{array}{r}+7\\ \underline{+5}\end{array}$

3. $\begin{array}{r}+8\\ \underline{-3}\end{array}$

4. $\begin{array}{r}-4\\ \underline{+7}\end{array}$

5. $\begin{array}{r}-4\\ \underline{-1}\end{array}$

6. $\begin{array}{r}+7\\ \underline{-4}\end{array}$

7. $\begin{array}{r}+9\\ \underline{+1}\end{array}$

8. $\begin{array}{r}-7\\ \underline{+7}\end{array}$

9. $\begin{array}{r}-7\\ \underline{-3}\end{array}$

10. $\begin{array}{r}0\\ \underline{-4}\end{array}$

11. $\begin{array}{r}-4\\ \underline{0}\end{array}$

12. $\begin{array}{r}-6\\ \underline{-9}\end{array}$

13. $\begin{array}{r}+7\\ \underline{+9}\end{array}$

14. $\begin{array}{r}0\\ \underline{-7}\end{array}$

15. $\begin{array}{r}+6\\ \underline{-6}\end{array}$

16. $\begin{array}{r}+2\\ \underline{0}\end{array}$

17. $\begin{array}{r}-8\\ \underline{+2}\end{array}$

18. $\begin{array}{r}-9\\ \underline{-3}\end{array}$

19. $\begin{array}{r}+1\\ \underline{+8}\end{array}$

20. $\begin{array}{r}-9\\ \underline{0}\end{array}$

21. $\begin{array}{r}0\\ \underline{-1}\end{array}$

22. $\begin{array}{r}-5\\ \underline{-5}\end{array}$

23. $\begin{array}{r}-1\\ \underline{+9}\end{array}$

24. $\begin{array}{r}-7\\ \underline{0}\end{array}$

25. $\begin{array}{r}+5\\ \underline{-3}\end{array}$

Find the following quotients:

26. $\frac{48}{8}$.

27. $\frac{24}{-6}$.

28. $\frac{-55}{-11}$.

29. $\frac{-75}{15}$.

30. $\frac{63}{-21}$.

31. $\frac{56}{-7}$.

32. $\frac{32}{4}$.

33. $\frac{-2}{+2}$.

34. $\frac{-15}{3}$.

35. $\frac{-1}{-1}$.

36. $\frac{-70}{-10}$.
37. $\frac{-18}{2}$.
38. $\frac{63}{7}$.
39. $\frac{60}{-12}$.
40. $\frac{0}{17}$.
41. $\frac{-96}{4}$.
42. $\frac{66}{-6}$.
43. $\frac{-48}{-8}$.
44. $\frac{72}{9}$.
45. $\frac{42}{-6}$.
46. $\frac{4.2}{-2}$.
47. $\frac{-8.1}{9}$.
48. $\frac{-9.6}{4}$.
49. $\frac{3}{-3}$.
50. $\frac{-36}{1.2}$.

89. Summary. In this chapter you have learned

(1) The meaning and important uses of positive and negative numbers.
(2) How to picture signed numbers.
(3) How to add signed numbers.
(4) How to subtract one signed number from another.
(5) How to multiply one signed number by another.
(6) How to divide one signed number by another.

Test on Chapter VII

1. Find the sum, the difference, the product, and the quotient (upper number divided by lower number) in each of the following:

(1) $\begin{array}{r}+12\\-4\end{array}$ (2) $\begin{array}{r}-8\\-16\end{array}$ (3) $\begin{array}{r}-24\\+6\end{array}$ (4) $\begin{array}{r}12\\+8\end{array}$

2. Perform the indicated operations in each of the following:

(1) $(-15)+(+5)$.
(2) $(-12)+(-24)$.
(3) $(-10)-(+5)$.
(4) $(-25)-(-10)$.
(5) $(+8)\times(-4)$.
(6) $(-12)\times(-5)$.
(7) $(+300)\div(-20)$.
(8) $(+20)+(-8)+(+2)$.
(9) $(-5)+(-15)+(+8)+(-3)$.
(10) $12-7+(-3)+5-(-10)$.
(11) $(-3)\times(+5)\times(-2)$.
(12) $(-4)\times(-\frac{1}{2})\times(+5)\times(-6)$.
(13) $(-3.6)\div(-1.2)$.
(14) $(+8)\times(-5)\div(-4)$.

3. The table given below states the prices of some well-known types of automobiles during a certain year. The average price of these cars was \$482. By using signed numbers, indicate how the price of each type of car differed from this average.

(1) Two-door Sedan	\$494	(4) Sport Coupé	\$550
(2) Four-door Sedan	\$570	(5) Coupé	\$498
(3) Roadster	\$385	(6) Phaëton	\$395

4. The monthly gains or losses from a certain business, for six successive months, were as follows:

+ \$1250, − \$750, − \$350, + \$2450, − \$1000, + \$5000.

(1) What was the total gain or loss for this period of time?

(2) What was the average gain or loss per month?

5. The following list suggests illustrations of signed numbers. Copy the numbers in the list and write after each a plus or a minus sign, according as the corresponding illustration refers to positive or negative numbers.

(1) Height above sea level.
(2) Deposits in a bank.
(3) A penalty in a football game.
(4) Motion upward.
(5) The forward motion of an airplane.
(6) North latitude.
(7) A drop in temperature.
(8) Income.
(9) Expenses.
(10) Dates B.C.
(11) A gain in a football game.
(12) The amount an airplane is retarded by wind.

CHAPTER VIII

FUNDAMENTAL OPERATIONS

1. ADDITION AND SUBTRACTION

90. The Fundamental Operations. The methods of combining simple algebraic expressions by the four fundamental operations, namely, addition, subtraction, multiplication, and division, and the use of signed numbers in these operations, have been discussed in previous chapters. In this chapter the application of these principles to more complicated algebraic expressions will be explained.

91. Important Terms Defined. A knowledge of the following definitions will be necessary in connection with the work of this chapter.

Monomial. An algebraic expression whose parts are not separated by the signs + or − is called a *monomial,* or a *term.*

In the preceding pages you have become familiar with *simple* terms such as $3h$, $4n^2$, $5ab$, $\frac{h}{4}$. Expressions such as $3(a+b)$ and $\frac{b+c}{2}$ are sometimes called *compound* terms.

Monomials which have the same literal factors are called *similar terms,* or *like terms.*

Thus, $4ab$ and $-6ab$ are similar terms.

Any one of the factors of a term is called the *coefficient* of the product of the remaining factors.

Thus, in the term $4xy$, 4 is the coefficient of xy, x is the coefficient of $4y$, and so on.

In a term like $6a^2b$, the factor 6 is called the *numerical coefficient* of a^2b. Ordinarily the word "coefficient" is used in the sense of *numerical* coefficient. In any term whose numerical coefficient is not expressed, the coefficient 1 is understood.

Thus, ac, bcd, and x^2y are the same as $1\,ac$, $1\,bcd$, and $1\,x^2y$.

Binomial. An expression which consists of two terms is called a *binomial.*

Thus, $x + 2y$, $a^2 - b^2$, $3n + 6$, and $\frac{a}{b} + c$ are binomials.

Trinomial. An expression containing three terms is called a *trinomial.*

Thus, $2a + b - 5$, $x + \frac{y}{3} - z$, and $2(a + b) - 3c + d$ are trinomials.

Polynomial. Any expression consisting of two or more terms is called a *polynomial.*

Hence observe that the term "polynomial" really includes both the binomial and the trinomial.

92. Addition of Monomials. By making use of facts that have already been learned, the following additions can be easily understood:

$$\begin{array}{rrrr} +5d & \qquad -5d & \qquad +5d & \qquad -5d \\ \underline{+2d} & \qquad \underline{-2d} & \qquad \underline{-2d} & \qquad \underline{+2d} \\ +7d & \qquad -7d & \qquad +3d & \qquad -3d \end{array}$$

From these illustrative examples we may infer the following rule:

To add similar terms, find the algebraic sum of their coefficients and multiply it by the common literal factor.

NOTE. Observe that this statement expresses in more general form the Rule for Combining Similar Terms explained on pages 14 and 15.

Exercises. *Adding Similar Terms*

Numbers 1 to 12, oral

Add in each of the following cases:

1. $$\begin{array}{r} 4\,n \\ \underline{14\,n} \end{array}$$

2. $$\begin{array}{r} 7\,w \\ \underline{-3\,w} \end{array}$$

3. $$\begin{array}{r} 18\,r \\ \underline{-12\,r} \end{array}$$

4. $$\begin{array}{r} -15\,c \\ \underline{5\,c} \end{array}$$

5. $$\begin{array}{r} 12\,a \\ \underline{-8\,a} \end{array}$$

6. $$\begin{array}{r} -32\,n \\ \underline{-17\,n} \end{array}$$

7. $$\begin{array}{r} -27\,a^2 \\ \underline{12\,a^2} \end{array}$$

8. $$\begin{array}{r} -3\,bh \\ \underline{-13\,bh} \end{array}$$

9. $$\begin{array}{r} -n^3 \\ \underline{9\,n^3} \end{array}$$

10. $$\begin{array}{r} 10\tfrac{2}{3}\,ab \\ \underline{5\tfrac{1}{3}\,ab} \end{array}$$

11. $$\begin{array}{r} -4\tfrac{3}{4}\,c \\ \underline{19\tfrac{1}{2}\,c} \end{array}$$

12. $$\begin{array}{r} 2.3\,b \\ \underline{-15.5\,b} \end{array}$$

13. $$\begin{array}{r} -24\,s \\ -12\,s \\ \underline{6\,s} \end{array}$$

14. $$\begin{array}{r} 75\,d^2 \\ -35\,d^2 \\ \underline{15\,d^2} \end{array}$$

15. $$\begin{array}{r} 7\,ab \\ -4\,ab \\ -5\,ab \\ \underline{ab} \end{array}$$

16. $$\begin{array}{r} -12\,e^3 \\ 8\,e^3 \\ -4\,e^3 \\ \underline{20\,e^3} \end{array}$$

17. $$\begin{array}{r} 19\,xy \\ 17\,xy \\ \underline{-20\,xy} \end{array}$$

Find the sum in each of the following cases:

18. $12\,n + (-4\,n)$.

19. $-15\,p + (+7\,p)$.

20. $-9\,s + (-3\,s)$.

21. $9\,n + (-3\,n) + (-2\,n)$.

22. $10\,c + (-12\,c) + (+c)$.

23. $-18\,s + (-2\,s) + (+5\,s)$.

24. $-11\,w + (+w) + (-7\,w)$.

25. $25\,w^2 + (-15\,w^2) + (+10\,w^2)$.

26. $12\,ab + (-3\,ab) + (+2\,ab)$.

27. $7\,n^3 + (-6\,n^3) + (-3\,n^3)$.

93. Subtraction of Monomials. As in the case of signed numbers, *a monomial may be subtracted from another monomial by changing the sign of the subtrahend and then proceeding as in addition.*

This is shown in the following illustrative examples:

Subtract:

$$\begin{array}{rrrr} (1)\ +8\,d & (2)\ -8\,d & (3)\ -8\,d & (4)\ +8\,d \\ \underline{+3\,d} & \underline{+3\,d} & \underline{-3\,d} & \underline{-3\,d} \end{array}$$

THINK:

$$\begin{array}{rrrr} +8\,d & -8\,d & -8\,d & +8\,d \\ \underline{\text{plus } -3\,d} & \underline{\text{plus } -3\,d} & \underline{\text{plus } +3\,d} & \underline{\text{plus } +3\,d} \\ +5\,d & -11\,d & -5\,d & +11\,d \end{array}$$

Exercises. *Subtraction of Monomials*

Numbers 1 to 12, oral

Subtract in each of the following cases:

1. $\begin{array}{r} 5\,d \\ \underline{3\,d} \end{array}$

2. $\begin{array}{r} 4\,w \\ \underline{7\,w} \end{array}$

3. $\begin{array}{r} 5\,n \\ \underline{-2\,n} \end{array}$

4. $\begin{array}{r} -6\,s \\ \underline{-4\,s} \end{array}$

5. $\begin{array}{r} -10\,p \\ \underline{0} \end{array}$

6. $\begin{array}{r} 5\,a \\ \underline{-10\,a} \end{array}$

7. $\begin{array}{r} -8\,b \\ \underline{b} \end{array}$

8. $\begin{array}{r} c \\ \underline{-2\,c} \end{array}$

9. $\begin{array}{r} 12\,n^2 \\ \underline{-2\,n^2} \end{array}$

10. $\begin{array}{r} -9\,e^3 \\ \underline{-e^3} \end{array}$

11. $\begin{array}{r} -4\,bh \\ \underline{-14\,bh} \end{array}$

12. $\begin{array}{r} 0 \\ \underline{-3\,n} \end{array}$

13. $\begin{array}{r} -4\frac{3}{4}\,d \\ \underline{-2\frac{1}{4}\,d} \end{array}$

14. $\begin{array}{r} 7\frac{2}{3}\,dw \\ \underline{10\frac{1}{3}\,dw} \end{array}$

15. $\begin{array}{r} -20.3\,r^2 \\ \underline{8.4\,r^2} \end{array}$

16. $\begin{array}{r} 4(d+b) \\ \underline{-2(d+b)} \end{array}$

Find the difference in each of the following cases:

17. $9\,a-(+5\,a)$. $+4a$

18. $7\,c-(+3\,c)$.

19. $(-2\,b)-(+10\,b)$.

20. $(-12\,n)-(-2\,n)$.

21. $(-20\,s^2)-(-12\,s^2)$.

22. $14\,bh-(-7\,bh)$.

23. $dw-(-4\,dw)$.

24. $(-5\,e^3)-(+e^3)$.

25. Subtract $-24\,r^2$ from $-6\,r^2$.

26. How much larger is $25\,d$ than $-10\,d$?

94. Double Use of the + and – Signs. In the preceding pages the + and – signs have been used in two different senses. They either indicate opposite qualities such as gain and loss, or they refer to the operations of addition and subtraction respectively. In any given case, it is usually not difficult to decide which interpretation is preferable. The following illustrative examples will show that either interpretation leads to the same result:

Illustrative Example 1. Find the sum of $+a$, $-b$, and $+c$.

The required sum might be written in the form

(1) $$(+a)+(-b)+(+c).$$

But we know that subtracting a signed number is the same as adding the corresponding opposite number. Hence the above expression might have been written

(2) $$(+a)-(+b)+(+c).$$

Since, however, a means $+a$, b means $+b$, and c means $+c$, the expression in step (2) may be written

(3) $$a-b+c.$$

Hence the signs which in the original example were *signs of opposition* have become *signs of operation.*

Illustrative Example 2. Combine $7n-3n+8n-10n$.

By reversing the process explained above, we may obtain the result in either one of two ways. Thus, we may write

(1) $$(+7n)-(+3n)+(+8n)-(+10n),$$

in which case the original signs are regarded as *signs of operation.*

But the expression (1) may also be written in the form

(2) $$(+7n)+(-3n)+(+8n)+(-10n),$$

in which case the original signs appear as *signs of opposition.*

You will note that the same result is obtained in both cases, namely, $+2n$. It is usually better, however, to follow the *second* plan; that is, *to consider the given expression as the sum of a series of signed numbers.*

Exercises. *Double Use of + and − Signs*

Numbers 1 to 8, oral

In each of the following exercises you may obtain the result by considering the signs either as signs of operation or as signs of opposition. Combine the similar terms in each case:

1. $10 - 3 + 5$.
2. $5 + 2 - 9 + 12 - 7$.
3. $8 - 4 + 11 + 6 - 1$.
4. $2a + 9a - 3a$.
5. $5d - 4d + 6d - d$.
6. $7p + p - 3p - 4p$.
7. $12bh - 3bh - bh + 5bh$.
8. $10n^2 + 5 - 7n^2 + n^2 - 2$.
9. $8 - 10 + 3$.
10. $-10 + 4 - 2 - 7$.
11. $6 - 9 - 3 + 12$.
12. $-9 + 2 - 7 - 11 + 5$.
13. $3c - 5c + 8c - 12c$.
14. $b - 9b - 4b + 12b - 3b$.
15. $7n^2 - 8n^2 + 20n^2 - n^2 - 6n^2$.
16. $-5ab + 2ab - 11ab + ab + 4ab$.
17. $3x - 5 + 7x + 3 - 5x - 10$.
18. $10n - 3n + 8 + 7n - 16 + 2n$.
19. $-s^2 + 9s^2 - 15 - 5s^2 - 12s^2 + 9$.
20. $5a - 7a + 3b - 3a - b + 2a$.

95. Addition of Polynomials. Suppose that a man can travel m miles an hour when walking and d miles an hour when riding. One day he walks for 2 hours and rides for 3 hours, thus covering a distance of $(2m + 3d)$ miles. Another day he walks for 3 hours and rides for 5 hours, thus covering a distance of $(3m + 5d)$ miles. To find the total number of miles he traveled during these two days, we may arrange our work *horizontally*, as follows:

$$t = 2m + 3d + 3m + 5d = 5m + 8d.$$

Instead, we might have used a *vertical* arrangement, as follows:

$$\begin{array}{r} 2m + 3d \\ \underline{3m + 5d} \\ 5m + 8d \end{array}$$

Observe that we have thus added two *binomials*, which have been so arranged that *the similar terms are written under each other*. In like manner, we may add any polynomials.

96. Checking. In all mathematical work it is desirable to apply some check in order to make sure that the result obtained is correct. The method of checking which is found most useful in the case of the fundamental operations of algebra is that of substituting for each letter appearing in the given expressions some convenient numerical value. Thus, to check the addition of the three polynomials shown in the following illustrative example, we may let

$$a = 4, \quad b = 3, \quad \text{and} \quad c = 2.$$

Operation		*Check*
$5a + 4b + c$	$=$	$20 + 12 + 2 = 34$
$2a - 5b + 6c$	$=$	$8 - 15 + 12 = 5$
$-3a + 6b - 8c$	$=$	$-12 + 18 - 16 = -10$
$4a + 5b - c$	$=$	$16 + 15 - 2 = 29$

Observe that the sum of the numerical values of the three polynomials is 29, and that the numerical value of the final result is also 29. Hence we may infer that the work is correct.

Such a check cannot be accepted as final, however, for it may happen that, in substituting, two errors have been made such that one just offsets the other. Since this is quite unlikely to happen, the process explained above ordinarily affords sufficiently reliable evidence that the work is correct.

In checking expressions involving squares or cubes by this method, care should be exercised that the required operations are performed in their proper order (see pages 26 and 46).

Exercises. *Adding Polynomials*

Numbers 1 to 8, oral

Add in each of the following cases:

1. $\begin{array}{r} 3a + 10b \\ 5a + \ \ 6b \\ \hline \end{array}$

2. $\begin{array}{r} a + 4b \\ 3a + 7b \\ \hline \end{array}$

3. $\begin{array}{r} -x + 12 \\ 4x - \ \ 5 \\ \hline \end{array}$

4. $\begin{array}{r} 7s^2 + 15t^2 \\ s^2 + \ \ 8t^2 \\ \hline \end{array}$

5. $\begin{array}{r} 3a + 2b - 5c \\ a - 5b - 3c \\ \hline \end{array}$

6. $\begin{array}{r} x^2 + 6x + 9 \\ 3x^2 + 2x + 1 \\ \hline \end{array}$

7. $\begin{array}{r} 4a^2 - 34ab + \ \ b^2 \\ 2a^2 + \ \ 6ab - 3b^2 \\ \hline \end{array}$

8. $\begin{array}{r} n^2 - 3n + 5 \\ 5n^2 - 2n - 7 \\ -3n^2 + \ \ n + 2 \\ \hline \end{array}$

Add in each of the following cases, and check as explained on page 198:

9. $\begin{array}{r} 2a + 5b \\ 3a - 2b \\ \hline \end{array}$

10. $\begin{array}{r} 7a + 15 \\ -4a + \ \ 5 \\ \hline \end{array}$

11. $\begin{array}{r} -8n + 3a \\ 4n - \ \ a \\ \hline \end{array}$

12. $\begin{array}{r} 9x - 12y \\ -4x - \ \ 8y \\ \hline \end{array}$

13. $\begin{array}{r} -5x^2 + 11 \\ x^2 - \ \ 8 \\ \hline \end{array}$

14. $\begin{array}{r} 5a + 7b - 3c \\ a - 2b - 5c \\ \hline \end{array}$

15. $\begin{array}{r} x^2 - 4x + 10 \\ -2x^2 - 6x + \ \ 2 \\ \hline \end{array}$

16. $\begin{array}{rcrcr} 3x^2 & - & x & + & 5 \\ -x^2 & & & + & 7 \\ \hline \end{array}$

17. $\begin{array}{r} 2a^2 - 3ab + 4b^2 \\ 6a^2 - 2ab - 4b^2 \\ \hline \end{array}$

18. $\begin{array}{rcrcr} r^2 & + & 3rs & - & 5s^2 \\ 2r^2 & & & + & 7s^2 \\ -5r^2 & - & rs & - & 5s^2 \\ \hline \end{array}$

Note. Observe the arrangement of the terms in the preceding exercises. A similar plan should be used in the work in Exercises 19–24.

Find the sum of the polynomials given in each of the following cases, and check each result:

19. $4n + 3$, $n - 17$, $-2n + 8$.

20. $w^2 + 4w + 4$, $2w^2 - 3w + 5$, $-7w^2 + w - 12$.

21. $6h^2 - 3h + 15$, $3h^2 - 9$, $h + 3$.

22. $2x^2 - 5x + 7$, $x - 3x^2 - 8$, $9 + x^2 - 2x$.

23. $3a + 2b - 9c$, $c - 5b + a$, $7b - 6a + 2c$.

24. $10n^2 - 3n - 8$, $n - 4n^2$, $6 + n^2 - 5n$.

25. The dimensions of a rectangle are $2s + 1$ and $5s - 3$. A second rectangle has the dimensions $3s - 4$ and $10s - 25$.

(1) Find the perimeter of each rectangle.

(2) Find the sum of these perimeters.

97. Subtraction of Polynomials. In the subtraction of polynomials the work may be arranged as in addition.

For example, subtract $7a^2 - 5a - 3$ from $5a^2 + 7 - 3a$, and check the result.

$$\begin{array}{rcl} \textit{Operation} & & \textit{Check.}\ \text{Let } a = 3. \\ 5a^2 - 3a + 7 & = & 45 - 9 + 7 = +43 \\ \underline{7a^2 - 5a - 3} & = & 63 - 15 - 3 = \underline{+45} \\ -2a^2 + 2a + 10 & = & -18 + 6 + 10 = -2 \end{array}$$

Thus the problem becomes one of subtracting similar terms which have been written under each other.

Checking. The check might have been performed as in arithmetic by adding the resulting difference to the subtrahend and thus obtaining a sum equal to the minuend. The substitution check is used, however, since it gives a uniform method of checking for all operations.

NOTE. If the given polynomials contain squares or cubes, the number 1 should not be used in checking, since it will not detect a mistake in an exponent. For instance, if by mistake $-2a^4$ had been written for $-2a^2$ in the above result, either term would have the value -2 when we let $a = 1$. Hence the result will appear to check in either case. If, however, another number is substituted for a, the result will not check when the mistake $-2a^4$ has been made. Thus, when $a = 3$, we have $-2a^4 = -162$, while $-2a^2 = -18$, and in this case the result will check only for the value of the correct term.

Exercises. *Subtraction of Polynomials*

Numbers 1 to 10, oral

Subtract in each of the following cases:

1. $\begin{array}{r} 5a + 9b \\ \underline{7a + 3b} \end{array}$

2. $\begin{array}{r} 3a - 5 \\ \underline{-7a + 4} \end{array}$

3. $\begin{array}{r} -2b - 12c \\ \underline{3b - 7c} \end{array}$

4. $\begin{array}{r} 6x + y \\ \underline{2x + 3y} \end{array}$

5. $\begin{array}{r} -4x^2 + 14 \\ \underline{x^2 - 7} \end{array}$

6. $\begin{array}{r} 8e^3 - 27 \\ \underline{-4e^3 } \end{array}$

7. $\begin{array}{r} 6bh \\ \underline{-2bh + 4} \end{array}$

8. $\begin{array}{r} 7d - 5w + 3h \\ \underline{2d - 5w - 2h} \end{array}$

9. $\begin{array}{r} a^2 - 4a + 4 \\ \underline{-2a^2 - 8a + 14} \end{array}$

10. $\begin{array}{l} 0 \\ \underline{3n^2 - 6n + 9} \end{array}$

11. $\begin{array}{r} 3dw + 2wh - 4 \\ \underline{-4wh + 8} \end{array}$

12. $\begin{array}{r} 5\frac{1}{2}a^2 - 7ab - 3\frac{1}{3}b^2 \\ \underline{2\frac{1}{2}a^2 + 3\frac{2}{3}ab - 1\frac{2}{3}b^2} \end{array}$

13. $\begin{array}{r} 5.7x^2 - 3x + 1.2 \\ \underline{x^2 + .8} \end{array}$

14. $\begin{array}{r} 8e^3 + e - 5 \\ \underline{-4e^2 + 6e - 5} \end{array}$

Perform the indicated subtractions and check each result:

15. Subtract $12a - 3b + c$ from $-8a + 3b - 2c$.

16. Subtract $-4p - 3r + 5t$ from $-4p + 6r - t$.

17. From $2r^2 + 7rs - s^2$ subtract $r^2 - 3rs + 2s^2$.

18. Subtract $7bh + 2b - h$ from $bh - 2b + 5h$.

19. Subtract $-3a - 11b + 7$ from 0.

20. Subtract $5x^2 - 2x$ from $2x^2 - 9x - 1$.

21. From $-15w^2 - 8wd + d^2$ subtract $5w^2 - 10d^2$.

22. How much larger is $c^2 - 12c + 20$ than $-5c^2 + c - 8$?

23. By how much does $e^2 - 15e + 2$ exceed $e^2 + 5e - 2$?

24. How much smaller is $-4A + 2B - 14C$ than $A - 6B - 7C$?

25. Subtract $3x^2 - 5x + 2$ from $4x^3 - 7x^2 + 6x - 3$.

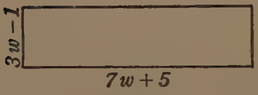


FIG. 1

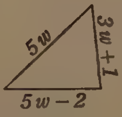


FIG. 2

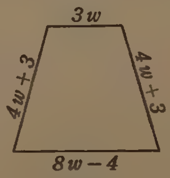


FIG. 3

26. How much larger is the perimeter of Fig. 1 than that of Fig. 2?

27. Subtract the perimeter of Fig. 2 from that of Fig. 3.

28. Subtract the perimeter of Fig. 3 from that of Fig. 1.

29. Subtract the perimeter of Fig. 2 from the sum of the perimeters of Figs. 1 and 3.

2. MULTIPLICATION

98. Rule of Signs for Multiplication. You have learned that in multiplication the product of two numbers having like signs is *positive*, whereas the product of two numbers having unlike signs is *negative*. This rule may be applied to literal as well as to numerical expressions.

Thus,
$$(+a) \times (+b) = +ab,$$
$$(-a) \times (-b) = +ab,$$
$$(+a) \times (-b) = -ab,$$
and
$$(-a) \times (+b) = -ab.$$

NOTE. The *rules* of algebra are often called its *laws*. Thus the *Rule of Signs* may also be referred to as the *Law of Signs*.

99. Rule of Order for Multiplication. The volume of a box whose dimensions are 2 feet, 3 feet, and 5 feet may evidently be found in several ways. Thus,

$$V = 2 \times 3 \times 5 = 2 \times 5 \times 3 = 3 \times 2 \times 5, \text{ and so on.}$$

Hence we may infer the following Rule of Order:

In multiplication, the factors may be arranged in any order.

That is, in general form, $a \times b = b \times a$.

100. Rule of Grouping for Multiplication. To obtain the result from any one of the products indicated above, such as $2 \times 5 \times 3$, we may first multiply 2×5 and then multiply the product by 3, or we may multiply 2 by the product of 5 and 3. For example,

$$2 \times 5 \times 3 = (2 \times 5) \times 3 = 2 \times (5 \times 3).$$

Hence we may infer the following Rule of Grouping:

In multiplication, the factors may be grouped in any convenient manner.

That is, in general form,

$$a \times b \times c = (a \times b) \times c = a \times (b \times c).$$

Exercises. *Applying the Rules of Order and Grouping*

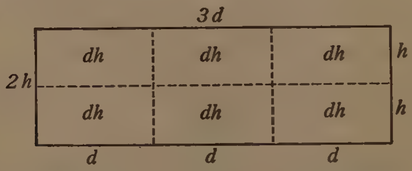
Find rapidly the following products:

1. $2 \times 14 \times 5$.
2. $13 \times 10 \times 3$.
3. $5 \times 7 \times 4 \times 5$.
4. $8 \times 5 \times 125$.
5. $16 \times 7 \times \frac{1}{2}$.
6. $15 \times 14 \times \frac{1}{3} \times 4$.
7. $12.5 \times 7 \times 8 \times 3$.
8. $1\frac{1}{4} \times 12 \times \frac{2}{3} \times 16$.

9. In finding the product of 7, 5, 8, and 2, there are 24 ways of arranging the factors. Write as many of these arrangements as you can find.

10. Find the area of a rectangle whose dimensions are $3\,d$ and $2\,h$.

Solution. From the diagram below it will be seen that the area is $6\,dh$.



The algebraic solution involves both the Law of Order and the Law of Grouping, as the following development will show:

$$A = 3\,d \times 2\,h.$$

This may be written $A = 3 \cdot d \cdot 2 \cdot h.$

Applying the Rule of Order, we have

$$A = 3 \cdot 2 \cdot d \cdot h.$$

Applying the Rule of Grouping, we have

$$A = (3 \cdot 2) \cdot (d \cdot h).$$

Hence $A = 6\,dh.$

11. Find the area of a rectangle whose dimensions are (1) $5\,l$ and $3\,w$; (2) $9\,b$ and $4\,a$; (3) $20\,d$ and $\frac{1}{2}\,h$.

Find the following indicated products:

12. $3 \times 4\,h$.
13. $5\,d \times 2$.
14. $8\,a \times 4$.
15. $2\,a^2 \cdot 6$.
16. $5 \cdot 4\,b \cdot 2$.
17. $2\,a \cdot 4\,b$.
18. $3\,a \cdot 5\,a$.
19. $6\,x \cdot 5 \cdot 2\,y$.
20. $2\,a \cdot 3\,b \cdot 4\,c$.
21. $5\,n \cdot 4\,n \cdot 2\,n$.
22. $2\,x \cdot (-5\,y)$.
23. $(-3\,a) \cdot (-2\,b)$.
24. $(-12\,h) \cdot \frac{1}{2}\,d$.
25. $8\,b \cdot (-2\,b)$.
26. $3\,w \cdot (-7\,w) \cdot 4$.

101. Rule of Exponents for Multiplication. You have learned that

$$a \times a = a^2,$$

and that

$$a \times a \times a = a^3.$$

In like manner, $a \times a \times a \times a \times a = a^5$, which is read "$a$ to the fifth power." In the term a^5 the small number 5 is called an *exponent*. It indicates how many times a has been used as a factor. The factor a which is repeated is called the *base*. The result a^5 is called the *power*. Similarly, in the expression $2^4 = 16$, 2 is the *base*, 4 is the *exponent*, and 2^4, or 16, is the *power*.

When no exponent is written with a number or letter, the exponent 1 is understood. Thus, a means a^1, and $3\,dw^2$ means $3^1 \cdot d^1 \cdot w^2$.

Accordingly, to find the product of $a^3 \times a^2$, we have

$$a^3 \times a^2 = (a \cdot a \cdot a) \times (a \cdot a) = a \cdot a \cdot a \cdot a \cdot a = a^5.$$

In brief form, $a^3 \times a^2 = a^{3+2} = a^5.$

Hence *the exponent of the product of powers having like bases is the sum of the original exponents.*

That is, in general form, $a^m \times a^n = a^{m+n}$.

Likewise, $(a^2)^3 = a^2 \cdot a^2 \cdot a^2 = a^{2+2+2} = a^6,$

or, more briefly, $(a^2)^3 = a^{2\times 3} = a^6.$

In general, $(a^m)^n = a^{m\times n}.$

In like manner,

$$(ab)^2 = ab \times ab = a \cdot a \cdot b \cdot b = a^2b^2,$$

or, more briefly, $(ab)^2 = a^2b^2.$

In general, $(ab)^n = a^nb^n.$

NOTE. Distinguish carefully between a coefficient and an exponent. For instance, in $4\,h$, 4 is a coefficient; in h^4, 4 is an exponent. The expression $4\,h$ means $h + h + h + h$, while h^4 means $h \cdot h \cdot h \cdot h$.

Exercises. *Exponents*

1. What is the meaning of x^4? b^3? a^7? a^5?

2. Using exponents, write each of the following expressions in brief form:

(1) $n \cdot n \cdot n \cdot n$.
(2) $s \cdot s \cdot s \cdot s \cdot s \cdot s$.
(3) $a \cdot a \cdot a \cdot a \cdot a$.
(4) $x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x$.
(5) $2 \times 2 \times 2 \times 2$.
(6) $10 \times 10 \times 10$.

3. Distinguish between $5\,n$ and n^5.

4. Write, in brief form, $b + b + b$; also $b \cdot b \cdot b$.

5. Using coefficients and exponents, write in briefest form each of the following expressions:

(1) $x + x + x + x$.
(2) $yy + yy + yy$.
(3) $3\,aaa - 2\,aaa$.
(4) $2\,bb + 5\,bb - 3\,bb$.
(5) $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$.
(6) $5 \cdot 5 \cdot 5 \cdot 5$.
(7) $5\,rrr + 3\,rr$.
(8) $5\,rrr \times 3\,rr$.

6. What is another way of writing $(ab)^2$? $(mn)^3$? $(xy)^5$?

7. Express as a power of 2 each of the following numbers:

4, 8, 16, 32, 64, 256.

8. Express as a power of 5 each of the following numbers:

25, 125, 625.

9. Write one million as a power of 10.

Find the following indicated products:

10. $a^4 \cdot a^3$.
11. $n^2 \cdot n^7$.
12. $p^5 \cdot p^3$.
13. $x^8 \cdot x^4$.
14. $s^5 \cdot s^6$.
15. $n^3 \cdot n$.
16. $x^2 \cdot x^4$.
17. $a \cdot a^5$.
18. $b^4 \cdot b^2 \cdot b^3$.
19. $x^3 \cdot x \cdot x^2$.
20. $R^2 \cdot R^3$.
21. $(-d) \cdot d^4$.
22. $p^{10} \cdot (-p^5)$.
23. $(-n^9) \cdot (-n^{11})$.
24. $r^6 \cdot (-r^3) \cdot r^9$.

102. Multiplication of Monomials. The multiplication of monomials involves the rules of signs, order, grouping, and exponents, which have already been considered.

Thus,
$$\begin{aligned}(+3a^2b)\times(-2a^3b^2c) &= (+3)\cdot(a^2b)\times(-2)\cdot(a^3b^2c)\\ &= (+3)\cdot(-2)\cdot a^2\cdot a^3\cdot b\cdot b^2\cdot c\\ &= -6a^5b^3c.\end{aligned}$$

Hence we may use the following method of finding the product of two or more monomials:

1. *Find the product of the coefficients, applying the Rule of Signs for Multiplication.*
2. *Multiply this result by the product of the literal factors, giving to each letter in the product an exponent equal to the sum of its exponents in the separate factors.*

Exercises. *Multiplication of Monomials*

REMEMBER THE KEY WORDS: **signs, order, grouping, exponents, and their abbreviations, S., O., G., E.**

Numbers 1 to 10, oral

Perform the indicated operations in each of the following cases:

1. $2w^3\cdot 3w^5$.
2. $3d^2\cdot d^4$.
3. $n^4\cdot(-n^3)$.
4. $(-8r^2)(-3r^5)$.
5. $(-5d)^3$.
6. $(-2a^2b^3)(6a^2b)$.
7. $(5D)(-3D^3)(-2D^2)$.
8. $(-b)(-10b^2)$.
9. $5a^7\cdot(-10a^2)$.
10. $-(2p)^3$.
11. $(-10R^2)(-10R^2)$.
12. $5y(-3x^2y)$.
13. $(-3s^3t)^2$.
14. $-(3s^3t)^2$.
15. $c(8c^3)(-\frac{3}{4}c^5)$.
16. $(-2nd)(-3n)(-4d^2)$.

17. Find the square of each of the following:

$5, -4, n^4, 3a^3, -7d, -6b^5, a^2b, 5x^3y^2.$

18. Find the cube of each of the following:

$3, -2, c^2, 4n^3, -3s, +5d^4, -x^2y^6, 2cd^5.$

19. Show that the following statements are true:

(1) $2^4 = 4^2.$
(2) $8^2 = 4^3.$
(3) $2^3 \cdot 2^4 = 2^7.$
(4) $3^2 \cdot 3^3 = 3^5.$
(5) $10^3 \cdot 10 = 10^4.$
(6) $(2^3)^2 = 2^6.$

EXPLANATION. *Order of Operations.* The directions given previously concerning the evaluation of algebraic expressions may now be extended to include any indicated power. The operations should be performed in the following order:

1. *Find the numerical value of any indicated power.*
2. *Perform the indicated multiplications or divisions in the order in which they occur.*
3. *Perform the indicated additions or subtractions.*

When a parenthesis occurs in the original expression, the operations indicated within the parenthesis should be performed first.

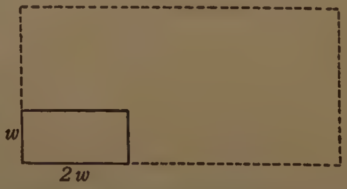
20. Find the numerical value of the following if $w = 2$ and $d = 3$:

(1) $5w^2 - 6d.$
(2) $10wd^2.$
(3) $4w^3d^2.$
(4) $(2w)^3 - 2d^2.$
(5) $w - (-2d)^2.$
(6) $5w^3d.$
(7) $(-3w^3)^2 + 2wd^2.$
(8) $5w^2d - (-3d)^3.$
(9) $6(w^3d)^2 + 5(-w)^5.$

21. The length of a rectangle is twice its width w.

(1) What is its perimeter? its area?

(2) If each side is made three times as long as in the original rectangle, what is the resulting perimeter? the resulting area?



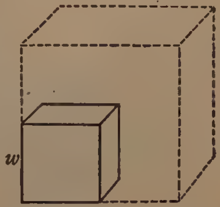
(3) Does the area become three times as large when each side is made three times as long?

22. A box in the form of a cube has an edge whose length is w.

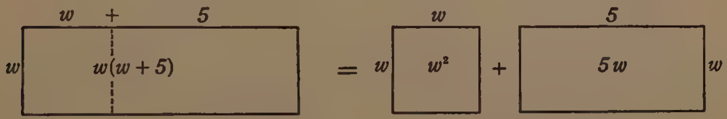
(1) What is the total length of its edges? the total surface? the volume?

(2) If each edge is doubled, what is the total surface of the resulting solid? the volume?

(3) Does doubling the edge double the surface? the volume?



103. Multiplication of a Polynomial by a Monomial. Suppose that the width of a rectangle is w, while the length is $w + 5$. To find the area of this rectangle, it becomes necessary to multiply w by $w + 5$. The diagram shows that the product is $w^2 + 5w$.



That is, $$w(w + 5) = w^2 + 5w.$$

Similarly, $$2a(a - 2b + 3c) = 2a^2 - 4ab + 6ac.$$

From these illustrations we infer the following rule:

> *To multiply a polynomial by a monomial, multiply each term of the polynomial by the monomial, uniting the resulting terms by their proper signs.*

That is, $$n(a + b - c) = na + nb - nc.$$

This rule is frequently called the *Rule of Distribution for Multiplication.*

In applying this rule, the work may be arranged either *horizontally* or *vertically*, as shown on the next page.

Horizontal Arrangement

$$-3a(a^2 - 2a + 1) = -3a^3 + 6a^2 - 3a.$$

Check. Let $a = 2$.

$$-6(4 - 4 + 1) = -24 + 24 - 6.$$
$$-6(1) = -6.$$
$$-6 = -6.$$

Vertical Arrangement *Check.* Let $a = 2$.

$$\begin{array}{rcr} a^2 - 2a + 1 & = & 4 - 4 + 1 = +1 \\ -3a & = & -6 \\ \hline -3a^3 + 6a^2 - 3a & = & -24 + 24 - 6 = -6 \end{array}$$

Exercises. *Multiplying Polynomials by Monomials*

Numbers 1 to 10, oral

Find the indicated products:

1. $3(a + b)$.
2. $5(d - 2w)$.
3. $2n(3x - y)$.
4. $-4r(r^2 + 2r - 3)$.
5. $6(a - 3b + 5c)$.
6. $b(3r - 2b)$.
7. $-2(-x + 3y - 6)$.
8. $5a(2b - 6c)$.
9. $-3p(p^2 - 6p)$. $= -3p^3 + 15p^2$
10. $n^2(3n^3 + 2n - 1)$.

Find the indicated products and illustrate each by a drawing:

11. $d(d + 3)$. 12. $h(2h + d + 4)$. 13. $2h(h + 2w + 3d)$.

Find the product in each of the following cases, and check each result:

14. $y^2(2y - 3)$.

15. $2n^2(n^2 + 3n - 1)$.

16. $-a^3(a^2 - 3a + 5)$.

17. $-h(1 - 12h + 32h^2)$.

18. $\begin{array}{r} 5a - 3b + 7 \\ 3a \\ \hline \end{array}$

19. $\begin{array}{r} n^2 + 5n - 8 \\ -2n^2 \\ \hline \end{array}$

20. $\begin{array}{r} 8y^3 + 4y^2 - y + 1 \\ 5y^2 \\ \hline \end{array}$

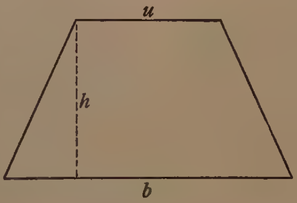
21. $\begin{array}{r} 7x^2 - 4xy + y^2 \\ -3x^2y \\ \hline \end{array}$

22. Find the area of a rectangle whose length is $4d - 5$ and whose width is $3d$.

23. The altitude of a rectangle is $5h$. What is its area if its base is $10h + 3$? if its base is $12h - 7$?

24. What is the volume of a rectangular solid whose height is $3b - 2$ and the area of whose base is $12b^2$?

25. The adjoining figure represents a *trapezoid*. The two parallel sides are called its *bases*. If the altitude of a trapezoid is denoted by h, the lower base by b, and the upper base by u, its area is found by the formula $A = \frac{h}{2}(b + u)$. Rewrite the formula, performing the indicated multiplication.



26. If the bases of a trapezoid are b and c respectively and its altitude is 4, what is its area?

27. Find the area of a trapezoid having an altitude of 6 and having as its bases w and 10 respectively.

28. An airplane traveled $(50d + 12)$ miles an hour for $3h$ hours. How far did it fly?

Find rapidly the following products:

29. 1001×17.

SUGGESTION. $1001 \times 17 = (1000 + 1)17 = 17{,}000 + 17 = 17{,}017$.

30. 105×16.

31. 112×6.

32. 204×8.

33. 1006×14.

34. 2003×21.

35. 99×13.

36. 999×16.

37. 197×3.

38. 99×101.

104. Removing Parentheses. In multiplying a polynomial by a monomial, we had occasion to remove the parentheses inclosing the given polynomial, by applying the Rule of Distribution. Since this process of removing parentheses is used very commonly in algebra, additional practice in dealing with parentheses is desirable. The following explanatory statements should be kept in mind:

1. *The removal of parentheses requires the careful application of the Rule of Signs for Multiplication.*

Thus, $3(a+b-c) = 3a+3b-3c.$

Similarly, $-2a(a^2-3a+5) = -2a^3+6a^2-10a.$

2. *Any parenthesis without an indicated coefficient may be considered as having a coefficient of 1.*

Thus, $(a+b) = 1(a+b) = a+b.$

Also, $-(x-2y) = -1(x-2y) = -x+2y.$

3. *The accepted order of operations requires an indicated multiplication to be performed before an addition or a subtraction.*

Thus, $5-3(a+2b) = 5-3a-6b.$

Hence $5-3(a+2b)$ *does not equal* $2(a+2b).$

4. *If one set of parentheses occurs within another, it is usually advisable to remove them one set at a time, starting with the innermost.*

Thus,

$$-3c[a+2b(b-c)] = -3c[a+2b^2-2bc] = -3ac-6b^2c+6bc^2.$$

Exercises. *Removing Parentheses*

Remove the parentheses and combine similar terms:

1. $5(2a+3)-7.$
2. $2(x-3)+5.$
3. $5n-4(n+7).$
4. $-3(5d-4)+12.$

5. $4a(2a + b - c) + 7ac$.

6. $3 + 5(2x - 9)$.

7. $3b - 2b(b - 1)$.

8. $11x(x - 4) + 7x$.

9. $-7a(-3 - 2a) + 4a$.

10. $16y^2 + 4y(3y - 5)$.

11. $5(3n - 4) + 6n$.

12. $-(2x - y) - 5y$.

13. $5b(-2m + b) + bm$.

14. $11a + 2(3b - c - a)$.

15. $8w(w^2 - 5w - 6) + 25w^2$.

16. $8 - 4(-2x + 3)$.

17. $6 - (s - 5)$.

18. $10w + w(6 - 4w)$.

19. $9n^2 - 6n + (1 - 4n + 4n^2)$.

20. $mx + 4(mx - 3) + 11$.

21. $7(5x - 3) - 6(3x + 4)$.

22. $3(5a + 1) + 2(3a - 5)$.

23. $5(x - y) - (2x + y)$.

24. $6s - 3(r + s + t) + (2r - s + 3t)$.

25. $5x - 7y + (2x - y)$.

26. $-(8d - 3) + (-3d + 4)$.

27. $6n^2 - (3n + 2) + 2n(n - 5)$.

28. $5[2n - 6(n + 2)] - 3n + 8$.

29. $3a[6 + 2(a - 3)] + 10a^2$.

30. $s\{s - 2(3 - 5s) + 7\} - 2s^2$.

31. $2x - 3[2x - 3(x - 5)]$.

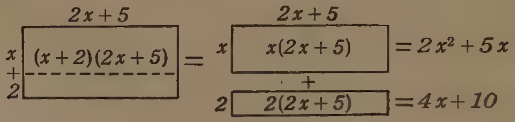
32. $4y + 2y[3y - (5 - y) + 6]$.

33. $8r - [7 - 2(3r - 5)]$.

34. $9x^2 + 3x + 2[2x - 5x(x - 1) - 10] - (2x^2 + 7x)$.

35. $(3x - 5) - \{6x + 4x(2x - 7) + 3x\} + 12$.

105. Multiplication of a Polynomial by a Polynomial. If the dimensions of a rectangle are $2x + 5$ and $x + 2$, its area is found by multiplying $2x + 5$ by $x + 2$. The diagram shows how this product may be obtained.



That is, $(x + 2) \times (2x + 5) = x(2x + 5) + 2(2x + 5)$
$= 2x^2 + 5x + 4x + 10 = 2x^2 + 9x + 10.$

Thus we see that the product is found by a repeated application of the Rule of Distribution. Observe that each term of the first binomial was multiplied by each term of the second binomial.

The work may also be arranged vertically, as follows:

Operation		**Check.** Let $x = 2$.
$2x + 5$	=	$4 + 5 = 9$
$x + 2$	=	$2 + 2 = 4$
$2x^2 + 5x$		36
$+ 4x + 10$		
$2x^2 + 9x + 10$	=	$8 + 18 + 10 = 36$

NOTE. A similar process is used in the arithmetical multiplication of numbers having two or more places, except that in algebra the work is carried out from left to right. For example, multiply 43 by 26 in the ordinary way, and then multiply (40 + 3) by (20 + 6) by the method explained above, comparing the corresponding steps.

106. Arrangement of Terms. In $4x^3 + 6x^2 + 5x + 7$, the terms are said to be arranged in *descending order*, since the exponents of the letter x descend, from left to right, in the order of magnitude. Similarly, the terms of $7 + 5x + 6x^2 + 4x^3$ are said to be arranged in *ascending order*. In the multiplication of polynomials, it is usually advisable to arrange the terms in either descending or ascending order, as shown in the following illustrative example:

DESCENDING ORDER		ASCENDING ORDER
$\begin{array}{l} -x^2+x+5 \\ \underline{-7x+3\quad\;\;} \end{array}$	or	$\begin{array}{l} 5+x-x^2 \\ \underline{3-7x\quad\;\;} \end{array}$

If more than one letter occurs in the given expressions, the terms may be arranged according to the powers of one particular letter. For example, $5x^3 - xy^2 + 2y^3 - 3x^2y$, when arranged according to the descending powers of x, would read $5x^3 - 3x^2y - xy^2 + 2y^3$.

From the preceding explanations and illustrative examples we infer the following rule for multiplying a polynomial by a polynomial:

1. *Arrange the terms according to the descending or ascending powers of the same letter.*
2. *Multiply each term of the multiplicand by each term of the multiplier, and add the partial products.*

Exercises. ***Multiplying a Polynomial by a Polynomial***

Multiply:

1. $x + 3$ by $x + 5$.
2. $x - 12$ by $x + 7$.
3. $x - 8$ by $x - 7$.
4. $n + 15$ by $n - 4$.
5. $2n + 3$ by $3n + 10$.
6. $12r + 3s$ by $r - 4s$.
7. $7x^2 + 2x$ by $x - 3$.
8. $3c^2 + 2d$ by $3c^2 + 2d$.
9. $2r^2 - 9s^2$ by $2r^2 + 9s^2$.
10. $3w^2 - 6w$ by $2w + 1$.

Multiply in each case and check the result:

11. $(x^2 - 3x + 2)(x + 3)$.
12. $(n - 5 + 2n^2)(n - 2)$.
13. $(5y^2 + 3 - y)(3y - 5)$.
14. $(3x^2 - 5x - 2)(2x + 7)$.

15. $(7 + 2n^2 - 3n)(5n - 9)$.

16. $(a + 5)(5a^2 - 8 + 3a)$.

17. $(3 - 10h)(6h - 4h^2 - 10)$.

18. $(a^2 + 2ab + b^2)(a + b)$.

19. $(c^2 + cd + d^2)(c - d)$.

20. $(r^2 - 2rs + 4s^2)(3r - s)$.

21. $(t^2 - 3s^2 + 4st)(s - 2t)$.

22. $(3x - 5 + x^2)(3x^2 + 7 - 2x)$.

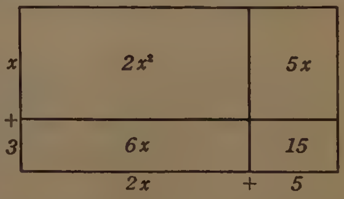
23. $(2a^2 + 3 - 6a)(5a - a^2 - 4)$.

24. $(4n + 3n^2 - 2)(n^2 - 6 - 2n)$.

25. $(3d - 1)(2d + 2)(d - 1)$.

26. $(x^3 - 2x^2 + 8 + 4x)(x - 2)$.

27. The adjoining figure shows a rectangle whose dimensions are $x + 3$ and $2x + 5$.



(1) Find the area by actual multiplication.

(2) Check your result by means of the diagram. The sum of the areas of the four small rectangles equals the total area of the large rectangle.

(3) What is the area if $x = 2$? if $x = 5$? if $x = 10$?

28. The dimensions of a closed rectangular box are represented by $w + 1$, $2w + 7$, and $3w - 2$.

(1) Find the expression which represents its total surface.

(2) Find its volume.

29. The width of a rectangular coal bin is 1 foot less than its height, and its length is 3 feet less than twice its height.

(1) If h represents its height, what is its volume?

(2) If the bin is 9 feet long, what are the other dimensions? what is its volume?

30. If a train travels at the rate of $(2r + 5)$ miles per hour for $(n - 3)$ hours, what distance will it cover?

31. If a grocer sells $(6p - 1)$ pounds of coffee at $(c - 2)$ cents per pound, and $(5p + 3)$ pounds of tea at $(2c - 3)$ cents per pound, what is the total amount received?

3. Division

107. Rule of Signs for Division. You have learned that in division the quotient of two numbers having like signs is positive, while the quotient of two numbers having unlike signs is negative. This rule may be applied to literal as well as numerical expressions. Thus,

$$\frac{+20d}{+5} = +4d. \qquad \frac{-20d}{+5} = -4d.$$

$$\frac{-20d}{-5} = +4d. \qquad \frac{+20d}{-5} = -4d.$$

108. Rule of Exponents for Division. Suppose that a^5 is to be divided by a^3. It is evident that the quotient must be an expression which, when multiplied by a^3, produces a^5. Hence, since $a^3 \times a^2 = a^5$, we see that $\frac{a^5}{a^3} = a^2$. This result might have been obtained as follows:

$$\frac{a^5}{a^3} = \frac{\cancel{a} \cdot \cancel{a} \cdot \cancel{a} \cdot a \cdot a}{\cancel{a} \cdot \cancel{a} \cdot \cancel{a}} = a^2.$$

Since three of the factors of a^5 are removed when we divide by a^3, we may write, more briefly,

$$\frac{a^5}{a^3} = a^{5-3} = a^2.$$

Hence, in the case of powers having like bases, *the exponent of the quotient may be found by subtracting the exponent of the divisor from that of the dividend.*

That is, in general form, $\frac{a^m}{a^n} = a^{m-n}.$

109. Division of Monomials. The division of one monomial by another involves the above rules of signs and exponents.

Thus, $\frac{-18\,b^3h^4}{+6\,b^2h} = -3\,bh^3$, since $(+6\,b^2h)\cdot(-3\,bh^3) = -18\,b^3h^4$.

Also, $\frac{-24\,r^2s^5t}{-3\,r^2s^2} = +8\,s^3t$, since $(-3\,r^2s^2)\cdot(+8\,s^3t) = -24\,r^2s^5t$.

Hence we may use the following method of dividing a monomial by a monomial:

1. *Find the quotient of the coefficients, applying the law of signs for division; then multiply this result by the quotient of the literal factors.*
2. *Find the quotient of the literal factors by subtracting the exponent of each letter in the divisor from that of the corresponding letter in the dividend, thus obtaining the exponent to be used with that letter in the quotient. Factors which have like bases and exponents cancel each other and give 1 as a factor of the quotient.*

Exercises. *Division of Monomials*

Numbers 1 to 10, oral

Divide:

1. $12\,d$ by (-2).
2. $-18\,a$ by (-6).
3. $-24\,n$ by 4.
4. $-28\,t^2$ by 4.
5. $40\,lwh$ by (-5).
6. $-42\,nr$ by (-6).
7. $-32\,b^3$ by 8.
8. n^6 by n^2.
9. d^8 by d^5.
10. nd^3 by $(-d^2)$.
11. $-w^5$ by w.
12. $-15\,dw$ by $(-3\,w)$.
13. m^3n^2 by $(-n^2)$.
14. $-8\,a^3b^4$ by $4\,ab^3$.
15. $24\,s^4$ by $(-6\,s^3)$.
16. $-60\,a^3b^2$ by $5\,a^2b^2$.

Perform the indicated division in each case:

17. $\dfrac{-36\,r^2s^5}{-12\,r^2s}$.

18. $\dfrac{54\,b^2h}{-6\,bh}$.

19. $\dfrac{-75\,p^6n^3}{15\,p^2n}$.

20. $\dfrac{-35\,b^2xy^3}{-7\,bxy^2}$.

21. $\dfrac{40\,r^5s^3t^2}{-5\,r^2t^2}$.

22. $\dfrac{48\,x^3y^2z}{-3\,x^2y^2}$.

23. Divide each of the following by $3\,a^2$:

$6\,a^3$, $12\,a^5$, $15\,a^2$, $-21\,a^4b$, $18\,a^2b^3$, $-33\,a^3b^2c$.

24. Divide each of the following by $-4\,rs^3$:

$8\,r^2s^3$, $-12\,r^5s^5$, $20\,rs^4$, $-36\,r^3s^3t$, $28\,rs^3$, $-4\,r^2s^6$.

25. Show that the following statements are true:

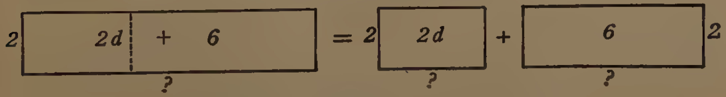
(1) $2^8 \div 2^3 = 2^5$.

(2) $3^5 \div 3^2 = 3^3$.

(3) $10^6 \div 10^4 = 10^2$.

(4) $5^3 \div 5^2 = 5$.

110. Division of a Polynomial by a Monomial. Suppose that the area of a rectangle is $2\,d + 6$, while its width is 2. To find the length of this rectangle it becomes necessary to divide $2\,d + 6$ by 2. The diagram shows that the result is $d + 3$, which may be obtained by considering the original figure as made up of rectangles having areas of $2\,d$ and 6 respectively, the width of each being 2.



That is, $\dfrac{2\,d+6}{2} = \dfrac{2\,d}{2} + \dfrac{6}{2} = d + 3.$

Similarly, $\dfrac{2\,d^3 - 4\,d^2 + 8\,d}{2\,d} = d^2 - 2\,d + 4.$

Such a procedure in division corresponds to the Rule of Distribution for Multiplication, and hence we may state the following rule:

> *To divide a polynomial by a monomial, divide each term of the polynomial by the monomial, uniting the resulting terms by their proper signs.*

That is, $\frac{na + nb - nc}{n} = \frac{na}{n} + \frac{nb}{n} - \frac{nc}{n} = a + b - c.$

Checking. Such examples may be checked either (1) by showing that *divisor* × *quotient* = *dividend*, or (2) by numerical substitution. The latter method is illustrated below.

Operation

$$\frac{-4n^3 - 12n^2 + 8n}{-4n} = n^2 + 3n - 2.$$

Check. Let $n = 2$.

$$\frac{-32 - 48 + 16}{-8} = 4 + 6 - 2.$$

$$\frac{-64}{-8} = 10 - 2.$$

$$8 = 8.$$

Exercises. *Dividing a Polynomial by a Monomial*

Numbers 1 to 7, oral

Divide:

1. $9n + 6$ by 3.
2. $4b + 12$ by 4.
3. $6n - 3$ by 3.
4. $n^2 - 2n$ by n.
5. $6n - 18n^2$ by $6n$.
6. $10a^2 - 25a$ by $(-5a)$.
7. $21x - 7$ by 7.
8. $27a^2 - 18a - 9$ by (-9).
9. $45x + 60x^2 - 20x^3$ by $5x$.
10. $12b^2 - 21bh + 18bh^2$ by $(-3b)$.

11. $3a^3 - 6a^2 + 18a$ by $3a$.

12. $9d^3 - 6d^2 + 3d$ by $3d$.

13. $x^4 - 3x^3 + x^2$ by x^2.

14. $4a^2 - 8ab + 4ab^2$ by $4a$.

Perform the indicated division in each case:

15. $\dfrac{6bh - 18b^2h - 12bh^2}{-6bh}$.

16. $\dfrac{-10x^3y + 6x^2y^2 - 2xy^3}{-2xy}$.

17. $\dfrac{3r^3s^3 - 5r^2s^2 + rs}{rs}$.

18. $\dfrac{6d^4b^3 - 12d^3b^2 - 9d^2b^3}{-3d^2b^2}$.

19. $\dfrac{16a^4b^2 - 24ab^3}{8ab^2}$.

20. $\dfrac{12m^5n^3 - 24m^4n^4 + 18m^2n^5}{6m^2n^3}$.

21. If a man's income for 12 days is $(36d - 24)$ dollars, how much does he earn per day?

22. An auto traveled $(150dh - 35h)$ miles in $5h$ hours. What was its hourly rate?

23. What is the length of a rectangle whose area is $4n^2 + 12n$ and whose width is $4n$?

24. The volume of a rectangular solid is $3h^3 - 6h^2 - 45h$. If its height is $3h$, what is the area of its base?

25. A rectangular solid has a base whose area is $5h^2$. If the volume is $5h^3 - 5h^2d + 15h^2$, find its altitude.

26. What is the altitude of a cylinder whose volume is $27\pi h + 45\pi - 9\pi d$ and the area of whose base is 9π?

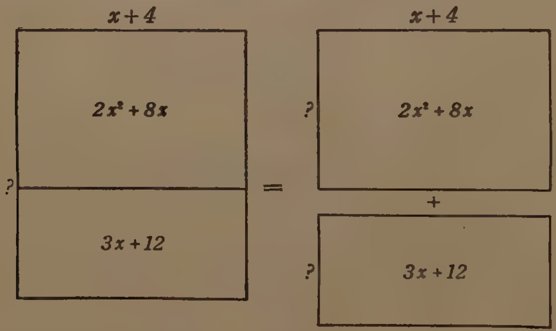
111. Division of a Polynomial by a Polynomial. It was shown previously that when $x + 4$ is multiplied by $2x + 3$, the process may be indicated as follows:

$$\begin{aligned}(2x+3)(x+4) &= 2x(x+4)+3(x+4)\\ &= 2x^2+8x+3x+12\\ &= 2x^2+11x+12.\end{aligned}$$

Hence it follows, from the principles of division, that

$$\frac{2x^2+11x+12}{x+4}=2x+3.$$

The actual process of division can readily be explained by means of a diagram such as the following:



Can you interpret the diagram? What is the width of the upper rectangle? How is it obtained? What is the width of the lower rectangle? Why?

The algebraic solution may be arranged in compact form as follows:

Operation

$$\begin{array}{rrl}
 & 2x+3 & \text{(Quotient)}\\
\text{(Divisor)}\ \underline{x+4} & \overline{)\,2x^2+11x+12} & \text{(Dividend)}\\
 & \underline{2x^2+\ \ 8x} & [2x(x+4)]\\
 & 3x+12 & \\
 & \underline{3x+12} & [3(x+4)]
\end{array}$$

Check

$$\begin{array}{r}
x+4\\
\underline{2x+3}\\
2x^2+\ \ 8x\\
\underline{+\ \ 3x+12}\\
2x^2+11x+12
\end{array}$$

Explanation. The first term of the dividend is the product of the first term of the divisor and the first term of the quotient. Hence the first term of the quotient, in the example above, is found by dividing $2x^2$ by x. The result is $2x$. This term represents the width of the upper rectangle in the diagram. The product of $2x$ and $x+4$ represents the first partial product, $2x^2 + 8x$. Subtracting this from the dividend, we obtain the remainder, $3x + 12$, which represents the area of the lower rectangle in the diagram. Dividing $3x$ by x, we obtain 3, the second term of the quotient, which represents the width of the lower rectangle. Multiplying 3 by $x + 4$, we obtain the second partial product, $3x + 12$, or the area of the lower rectangle. By subtracting this from $3x + 12$, we find that there is no remainder. In other words, the area of the large rectangle has been shown to be the product of $x + 4$ and $2x + 3$. The work may then be checked by actual multiplication.

112. Arrangement of Terms. As in the case of multiplication, the terms of both polynomials should be arranged according to the *descending* or the *ascending* powers of some common letter, before the division is begun. This is shown in the following illustrative example.

The checking may be done in either one of two ways; that is, by multiplication or by numerical substitution.

Illustrative Example. Divide $31a - 30a^2 + 8a^3 - 15$ by $2a - 5$.

Operation

$$
\begin{array}{rr}
 & \underline{4a^2 - 5a + 3 \qquad\qquad} \\
\underline{2a-5}\,\big| & 8a^3 - 30a^2 + 31a - 15 \\
 & \underline{8a^3 - 20a^2 \qquad\qquad\quad} \\
 & - 10a^2 + 31a \qquad\;\; \\
 & \underline{- 10a^2 + 25a \qquad\quad\;} \\
 & + \;\; 6a - 15 \\
 & \underline{+ \;\; 6a - 15}
\end{array}
$$

Check. Let $a = 2$.

$$\frac{\text{Dividend}}{\text{Divisor}} = \frac{64 - 120 + 62 - 15}{4 - 5}$$

$$= \frac{-9}{-1} = 9.$$

Quotient $= 16 - 10 + 3 = 9.$

Note. In checking by numerical substitution, care must be exercised not to substitute a value which will cause the divisor to become 0. Division by zero is meaningless in this work. Observe that $\frac{0}{5} = 0$, since $0 \times 5 = 0$. But the quotient $\frac{5}{0}$ calls for a number which, when multiplied by 0, produces 5. To avoid such contradictions, division by zero is excluded in both arithmetic and algebra.

The method to be used in dividing one polynomial by another may now be summarized as follows:

1. *Arrange the terms of both the dividend and the divisor according to the descending (or ascending) powers of the same letter.*
2. *Divide the first term of the dividend by the first term of the divisor, thus obtaining the first term of the quotient.*
3. *Multiply the divisor by the first term of the quotient and subtract this product from the dividend, thus obtaining a new dividend.*
4. *Repeat steps 2 and 3 as long as division is possible.*

This process is very similar to that of long division in arithmetic. Thus, divide 672 by 21 in the usual way. Then expand the work by dividing (600 + 70 + 2) by (20 + 1) by the method shown above, and compare the corresponding steps.

Exercises. *Dividing a Polynomial by a Polynomial*

Divide in each case and check the result:

1. $n^2 + 6n + 8$ by $n + 2$.
2. $d^2 + 3d + 2$ by $d + 1$.
3. $x^2 + 8x + 12$ by $x + 6$.
4. $x^2 + 7x + 12$ by $x + 4$.
5. $n^2 + 6n - 7$ by $n - 1$.
6. $b^2 + 6b - 16$ by $b - 2$.
7. $y^2 - 5y + 4$ by $y - 4$.
8. $a^2 - 3a - 10$ by $a - 5$.
9. $n^2 - 10n + 16$ by $n - 8$.
10. $w^2 - 5w - 24$ by $3 + w$.
11. $2x^2 + x - 6$ by $2x - 3$.
12. $6c^2 - 7c + 2$ by $3c - 2$.
13. $8n^2 - 22n + 15$ by $4n - 5$.
14. $20d - 8 + 12d^2$ by $8 + 4d$.
15. $6c^2 + 11cd - 10d^2$ by $3c - 2d$.

16. $4\,rs + 12\,r^2 - 5\,s^2$ by $5\,s + 6\,r$.

17. $15\,b^2 + 4\,a^2 - 23\,ab$ by $a - 5\,b$.

18. $4\,x^3 - 17\,x^2 - 16\,x + 5$ by $x - 5$.

19. $10\,p^3 + 13\,p^2 - 11\,p - 12$ by $2\,p + 3$.

20. $20\,a^3 - 15 - 22\,a^2 + 31\,a$ by $5\,a - 3$.

21. Divide $12\,x^3 - 5\,x^2 - 11\,x + 4$ by $3\,x - 2$, and check.

$$
\begin{array}{r}
4x^2+x-3\\
3x-2\,\big)\overline{12x^3-5x^2-11x+4}\\
\underline{12x^3-8x^2}\\
3x^2-11x\\
\underline{3x^2-2x}\\
-9x+4\\
\underline{-9x+6}\\
-2
\end{array}
\qquad \text{[Remainder]}
$$

NOTE. When the division is not exact we have, as in arithmetic, the relation

$$\frac{\text{Dividend}}{\text{Divisor}} = \text{Partial Quotient} + \frac{\text{Remainder}}{\text{Divisor}}.$$

Hence the complete result for the above division is $4\,x^2 + x - 3 + \dfrac{-2}{3\,x-2}$.

(Why cannot the fraction $\dfrac{-2}{3\,x-2}$ be written immediately after the quotient without using the + sign?)

Check. Let $x = 2$. Using the above relation, we have

$$\frac{96 - 20 - 22 + 4}{6 - 2} = 16 + 2 - 3 + \frac{-2}{6-2}.$$

$$\tfrac{58}{4} = 15 + (-\tfrac{1}{2}).$$

$$14\tfrac{1}{2} = 14\tfrac{1}{2}.$$

Divide and check:

22. $3\,x^2 + 11\,x + 8$ by $x + 3$.

23. $6\,n^2 - 7\,n - 8$ by $2\,n + 1$.

24. $13\,d - 14 + 12\,d^2$ by $3\,d - 2$.

25. $10a^2 - 31ab + 15b^2$ by $5a - 3b$.

26. $2x^3 - x^2 - 11x + 10$ by $2x + 5$.

27. $10c^2 + 3c + 3c^3 - 4$ by $c + 2$.

28. $10r - 8 + 9r^3 - 15r^2$ by $3r - 4$.

29. $4n^3 - 2n^2p - 4np^2 - 16p^3$ by $2n - 4p$.

30. $27ab^2 - 45a^2b + 12a^3 + 6b^3$ by $4a - 3b$.

31. $2x^3 - 3x^2 - 11x - 3$ by $x^2 - 3x - 1$.

32. $6c^4 + 10c^3 + 3c^2 + 21c - 5$ by $3c^2 - c + 5$.

33. $19a - a^3 - 42a^2 - 2 + 10a^4$ by $a + 2a^2 - 7$.

34. $2n^5 + n^4 - 12n^3 + 7n^2 - 19n + 15$ by $2n^2 + 5n - 4$.

35. $x^3 - 19x + 30$ by $x + 5$.

SUGGESTION. The dividend should be written $x^3 + 0x^2 - 19x + 30$.

36. $a^3 - 8$ by $a - 2$.

37. $27b^3 + 1$ by $3b + 1$.

38. $8r^3 - 27s^3$ by $2r - 3s$.

39. $3x^3 - 11x^2 - 16$ by $x - 4$.

40. $25n^4 + 71n^2 - 12$ by $5n + 2$.

113. Summary. In this chapter you have learned

(1) The meaning of *base*, *exponent*, and *power*.

(2) How to combine algebraic expressions by addition and subtraction.

(3) The laws of order, grouping, exponents, and distribution for multiplication.

(4) How to multiply a monomial by a monomial, a polynomial by a monomial, and a polynomial by a polynomial.

(5) The laws of exponents and distribution for division.

(6) How to divide a monomial by a monomial, a polynomial by a monomial, and a polynomial by a polynomial.

Supplementary Honor Work

Having Fun with Exponents and Large Numbers. The following exercises will extend your knowledge of exponents. It may surprise you to discover how rapidly the powers of ordinary numbers increase as the exponents take on successively the values 1, 2, 3, and so on. Thus, study and then extend the following table of the powers of 2:

$2^1 = 2$
$2^2 = 4$
$2^3 = 8$
$2^4 = 16$
$2^5 = 32$
$2^6 = 64$
$2^7 = 128$
$2^8 = 256$
$2^9 = 512$
$2^{10} = 1024$
$2^{11} = 2048$
$2^{12} =$ - - -
$2^{13} =$ - - -
$2^{14} =$ - - -
$2^{15} =$ - - -
$2^{16} =$ - - -
$2^{17} =$ - - -
$2^{18} =$ - - -
$2^{19} =$ - - -
$2^{20} =$ - - -

Exercises

1. Multiply 128 by 16, using the facts given in the above table.

Solution.

$$\begin{array}{r} 128 = 2^7 \\ \underline{16 = 2^4} \\ \text{Product} = 2^7 \times 2^4 = 2^{11} = 2048. \end{array}$$

(Explain.)

2. Using the method suggested in Exercise 1, find each of the following products:

(1) 64×16.
(2) 1024×32.
(3) 128×64.
(4) $256 \times 8 \times 16$.
(5) 128^2.
(6) 51.2×6.4.

3. Divide 1024 by 128, using the above table.

Solution. $$\frac{1024}{128} = \frac{2^{10}}{2^7} = 2^3 = 8.$$ (Explain.)

4. Using the method suggested in Exercise 3, find each of the following quotients:

(1) $128 \div 8$.
(2) $8192 \div 64$.
(3) $8192 \div 512$.
(4) $4096 \div 128$.
(5) $16{,}384 \div 512$.
(6) $2048 \div 32$.

5. The higher powers of 2 may be obtained by a skillful use of the Law of Exponents for Multiplication.

Thus, $2^{10} = 2^5 \times 2^5 = 32 \times 32 = 1024.$

Hence $2^{20} = 2^{10} \times 2^{10} = 1024 \times 1024 = ?$

Also, $2^{25} = 2^{20} \times 2^5 = ?$

Using this method, find 2^{50}; 2^{60}; 2^{64}.

6. Everyone has had 2 parents, 4 grandparents, 8 great-grandparents, and so on. Hence we find that, going back only one generation, each person has 2 ancestors; for 2 generations he has 2^2 ancestors; for 3 generations, 2^3 ancestors; and so on.

Allowing 30 years for each generation, there have been approximately 64 generations since the beginning of the Christian Era. For this period of time, how many ancestors have you had? Is this result possible? Where is the error?

7. At 4 per cent compound interest a sum of money is doubled in about 18 years. Using this approximate value, we find that in 18 years 1 dollar amounts to 2 dollars; in 36 years, to 4, or 2^2, dollars; in 54 years, to 8, or 2^3, dollars; and so on.

If a dollar had been put at 4 per cent compound interest by one of the Pilgrims in 1620, it would have doubled approximately 17 times up to the present time. How much money does this represent?

8. The story is told that many years ago the inventor of the game of chess presented the game to a great Oriental monarch. When the inventor was urged to ask for a favor, as a reward for his gift, he modestly requested that he be given 1 grain of wheat for the first square on the board, 2 grains for the second, 4 for the third, and so on, the

number of grains being doubled each time up to the 64th square. Hence the wise men of the kingdom added the numbers 1, 2, 4, 8, 16, and so on. They were amazed to find that the entire harvest of the kingdom would be insufficient to meet the inventor's "modest" request.

This will not surprise you if you learn that the required number of grains was $2^{64} - 1$. Allowing 9000 grains to a pint, the required number of bushels is said to exceed the present annual wheat production of the whole world. Can you verify this statement?

9. The powers of 10 are frequently used to shorten the writing of large numbers. Thus, $10^2 = 100$; $10^3 = 1000$; $10^4 = 10,000$; and so on. What is the meaning of 10^9? 10^{20}? Write one trillion as a power of 10.

10. When recording measurements which are very large or very small, scientists have found it convenient to make use of exponents. For example, the extremely small particles (called molecules) which make up every substance have been measured and found to have a diameter of $\frac{1}{10^8}$ cm. What part of a centimeter does this represent? Express one billionth of a centimeter in a similar manner.

Test on Chapter VIII

Find the sum and also the difference in each of the following cases:

$$1.\ \begin{array}{r} +8n \\ -5n \\ \hline \end{array} \qquad 2.\ \begin{array}{r} -\ \ 3d^2 \\ -10d^2 \\ \hline \end{array} \qquad 3.\ \begin{array}{r} -12bh \\ +20bh \\ \hline \end{array} \qquad 4.\ \begin{array}{r} +7h \\ h \\ \hline \end{array} \qquad 5.\ \begin{array}{r} -21s^3 \\ +\ \ 7s^3 \\ \hline \end{array}$$

$$6.\ \begin{array}{r} 3w - 7x + \ \ y \\ -5w - 3x + 8y \\ \hline \end{array} \qquad 7.\ \begin{array}{r} -7n^2 + 10n \qquad \\ n^2 - \ \ 4n - 12 \\ \hline \end{array}$$

Combine the terms in each of the following cases:

8. $2x^2 - 3 + 5x^2 - x + 9 - 3x - 4x^2$.

9. $-10ab^2 + 2a^2b - 3a^2b + 5ab^2 + 7a^2b$.

In each of the following exercises select the correct answer from those given in the parenthesis:

10. $(-3w) \times (-6w) = (-18w, -9w^2, \boxed{+18w^2)}$

11. $(-12d^3) \times d^2 = (-12d^5, -11d^5, -12d^6)$.

12. $6ab^2 \times 24a^2 = (30a^3b^2, 144a^2b^2, 144a^3b^2)$.

13. $(-28a^2b^5) \div (-4a^2b) = (-7a^2b^4, 7ab^4, 7b^4)$.

14. $d(1 - 10d) = (d - 10d^2, 1 - 10d^2, d - 10d)$.

15. $-2n^2(n^3 - 4n + 6) = (-2n^6 + 8n^2 - 12n^2, -2n^5 + 8n^3 - 12n^2, 2n^5 - 8n^3 - 12n^2)$.

16. $\dfrac{4r^3 - 8r^2 + 10r}{-2r} = (4r^3 - 8r^2 - 5, -2r^2 - 8r^2 + 10r, -2r^2 + 4r - 5)$.

Remove the parentheses and combine the resulting terms in Exercises 17–19:

17. $7 - 2(5a + 3)$.

18. $3y + (4x - 2y)$.

19. $4n(1 - n) + (n^2 + 3) - (2n - 1)$.

20. Multiply $2y^2 - 7y + 3$ by $y + 5$.

21. Divide $6c^3 - 5c^2 + 10c - 3$ by $3c - 1$.

22. The dimensions of a rectangle are w and $3w - 8$. A second rectangle has the dimensions $w + 4$ and $4w - 10$.

(1) Find the sum of the perimeters of these rectangles.

(2) Find the sum of the areas of these rectangles.

(3) By how much does the area of the second rectangle exceed that of the first?

Exercises. *Review*

1. Solve each of the following equations and check the result:

(1) $3x = 18$.
(2) $n + 9 = 19$.
(3) $2c - 5 = 15$.
(4) $\frac{a}{4} = 8$.
(5) $\frac{2a}{3} = 12$.
(6) $7p = 3p + 12$.
(7) $5n - 7 = n + 13$.
(8) $4d + 9 = 2d + 10$.
(9) $6s - 12 + s = 2s + 18$.
(10) $3b + 5 - b = 10 - 2b + 7$.

2. The formula $v = u + at$ is commonly used in physics.
(1) Find the value of u when $v = 45$, $a = 5$, and $t = 7$.
(2) Find the value of a when $v = 30$, $u = 3$, and $t = 10$.

3. The formula $g = .03264\,td^2$ is used in computing the discharge in gallons from a pump. Find g if $t = 80$ and $d = 18$.

4. A salesman's regular monthly salary, in dollars, is s. He also receives a commission of 3 dollars for each sale. Write a formula for his monthly income i, if the number of sales is represented by n.

5. The following table shows parcel-post rates in the seventh zone (1400 to 1800 miles) for certain weights.

Weight in pounds (p) =	1	2	4	7	10	12	15
Postage in cents (c) =	13	23	43	73	103	123	153

(1) Write a formula showing the relation of c and p.
(2) Using this formula, find c if $p = 11$; if $p = 20$.

6. Can you find three numbers whose sum is 34, such that the first two are consecutive and the third equals the sum of the first two?

7. Write a formula suggested by the following table:

If $x =$	1	3	6	8	12	15	20
then $y =$	8	14	23	29	41	50	65

8. The formula for the distance d which a person can travel in any number of hours h, if he travels at the rate of 16 miles an hour, is $d = 16\,h$.

(1) Draw a graph of this formula.

(2) From the graph find the distance covered in $5\frac{1}{2}$ hours.

(3) From the graph determine how many hours it will take to cover a distance of 128 miles.

9. Draw the graph of the formula $y = 4\,x$, using in succession the values $x = 0, 1, 2, 3, 4, 5, 6, 7$.

10. Draw the graph of the formula $y = 2\,x + 1$, using in succession the values $x = 0, 1, 2, 3, 4, 5, 6, 7$.

Multiply in each case and check the result:

11. $(9\,y - 2)(3\,y + 5)$.
12. $(4\,x + 1)(3\,x - 7)$.
13. $(xy - 6)(3\,xy + 7)$.
14. $(4\,a - 7\,b)(3\,a - 9\,b)$.
15. $(y - 7 + 5\,y^2)(3\,y - 2)$.
16. $(5\,x^3 - 2\,x + 4)(3\,x^2 - 4)$.
17. $(a^3 - 4\,a^2 + 5\,a - 2)(a^2 + 5\,a)$.
18. $(5\,n^3 + 9 - 2\,n^2 + 6\,n)(n - 3)$.
19. $(x^3 - 2\,x^2 + 7\,x - 3)(2\,x^2 + 4\,x - 5)$.
20. $(a^3 - 3\,a^2b + 4\,ab^2 - b^3)(a^2 - 2\,ab + b^2)$.

21. If $P = 2\,n^2 + 7\,n - 15$, $Q = 10\,n^2 - 29\,n + 21$, and $R = 2\,n - 3$, find

(1) $2\,P + 3\,Q - 4\,R$.
(2) $(P + Q) \div R$.
(3) $P \times Q$.
(4) $(P \div R) + (Q \div R)$.

CHAPTER IX

EQUATIONS OF THE FIRST DEGREE IN ONE UNKNOWN

114. Equations of the First Degree. An *equation of the first degree in one unknown* is one of such a form that

(1) the unknown appears only with the exponent 1;
(2) the unknown does not appear in any denominator.

For example, $3x + 7 = 9$, $15 = 7 - 2x$, and $4x + 5 = 9 + x$ are equations of the first degree.

Such equations are sometimes called *simple equations*, or *linear equations*. The reason for the latter name will be made clear in the following chapter.

An equation such as $4x^2 + 7x = 5$ is said to be of the *second degree*, since the term $4x^2$ contains the *second* power of x.

115. Short Methods of Solving. Only very simple equations of the first degree have been introduced thus far. The ready solution of more complicated equations of the first degree does not require any new principle. It does involve, however, a more *extended use* of these principles. Some of the most convenient special methods for solving such equations will be considered in this chapter.

116. Interchanging Members. It is evident that the two members of an equation may be interchanged without changing the value of the unknown number.

For example, the equation $7 = 5x - 3$ may be written in the more convenient form $5x - 3 = 7$.

Examine each of the following equations and observe how the solution is simplified by interchanging the members.

	ORIGINAL FORM	NEW FORM
(1)	$9 = a + 7.$	$a + 7 = 9.$
(2)	$20 = 4a - 4.$	$4a - 4 = 20.$
(3)	$8 = n - 7.$	$n - 7 = 8.$
(4)	$100 = 20x + 80.$	$20x + 80 = 100.$
(5)	$9 = 5x - 1.$	$5x - 1 = 9.$

117. Transforming an Equation by Addition or Subtraction. By a skillful use of the axioms of addition and subtraction a complicated equation can often be changed, or *transformed,* into an equation of a simpler type. How this is possible will become apparent from a study of the following statements:

$$\begin{array}{ll} (1) & a + 5 = b \\ & \quad\;\; 5 = 5 \\ \hline (2) & a \qquad = b - 5 \end{array} \qquad\qquad \begin{array}{ll} (1) & a - 5 = b \\ & \quad\;\; 5 = 5 \\ \hline (2) & a \qquad = b + 5 \end{array}$$

That is, if $a + 5 = b$, we have $a = b - 5$, by Axiom 2.

Also, if $a - 5 = b$, we have $a = b + 5$, by Axiom 1.

In the *first* case, the $+ 5$ of the first member disappears by subtraction, and $- 5$ appears in the second member. In the *second* case, the $- 5$ of the first member disappears by addition, and $+ 5$ appears in the second member.

Similarly, $7x - 5 = 9$ becomes $7x = 9 + 5$, or $7x = 14$.

Also, $9x = 4 + 7x$ becomes $9x - 7x = 4$, or $2x = 4$.

This short method of applying these axioms may be stated in the form of a *general rule,* as follows:

> *A term may be dropped from either member of an equation, provided the corresponding opposite term is written in the other member.*

That is, if $+7$ is dropped from the first member, we write -7 in the second member; if $-2x$ is dropped from the second member, we write $+2x$ in the first member; and so on.

In applying this rule, it *looks as if* the original term had been *moved* from one member of the equation to the other, with an opposite sign. Hence this rule is often called the *rule of transposition*, and is stated in the following form: *A term may be transposed from one member of an equation to the other, provided its sign is changed.*

The rule stated above will be called the *Rule of Transformation by Addition or Subtraction*, and the abbreviation Tr. may be used as a short way of referring to it.

Illustrative Example 1. Solve the equation

$$9n - 2 + 5n = 14n + 6 - 4n.$$

Solution. The equation may be transformed so that we obtain in the first member only terms containing n and in the second member only numerical terms.

Hence we have $9n + 5n - 14n + 4n = 6 + 2.$

By the Rule of Similar Terms, we have

$$4n = 8.$$

Hence $$n = 2.$$

Check. If 2 is substituted for n in the original equation, show that you obtain an identity.

Illustrative Example 2. Solve the equation

$$15x - 10 - 3x = 7x + 26 - x.$$

Solution. Using the method explained above, we have

(1)	$15x - 3x - 7x + x = 26 + 10.$	Tr.
(2)	$6x = 36.$	S. T.
(3)	$x = 6.$	D_6.

Check. Substitute 6 for x in the original equation.

NOTE. After the student has mastered this short method of solving equations, much of the work shown above may be performed mentally. For example, the similar terms in each member may be combined before the equation is transformed.

Exercises. *Applying the Rule of Transformation*

Numbers 1 to 10, oral

Solve each of the following equations by the method explained above. Interchange the members whenever it is convenient.

1. $4a - 2 = 3a$.
2. $7n - 10 = 5n$.
3. $2n + 14 = 7n + 9$.
4. $5x = 12 + x$.
5. $9a - 4 = 5a + 8$.
6. $8c + 9 = 5c - 6$.
7. $5x - 7 = 9 - 3x$.
8. $6a = 18 - 3a$.
9. $4p - 3 = 3p + 4$.
10. $d + 7 = 31 - 5d$.
11. $7w + 1 = w + 25$.
12. $12y - 5 = 34 - y$.
13. $5x - 2 + 3x = x + 12$.
14. $n + 3 + 2n = 23 - 2n$.
15. $8d - 1 = 6d + 1 - 2d$.
16. $3p - 8 + 4p = p + 28$.
17. $9w + 4 - 2w = 3w + 6$.
18. $7x + 15 - x - 3 = 36$.
19. $11s = 8s + 30 - 3s - 6$.
20. $12r - 15 = 9r + 9 - 3r$.
21. $6y - 3 + 2y = y + 7 + 2y$.
22. $10n + 6 - n = 31 - n$.
23. $4a - 3 = a + 18$.
24. $5t - 1.07 = 2.5 - 2t$.
25. $.7n - 5 = .4n + 7$.
26. $.06t - 12 = .04t + 18$.
27. $1.2y + 1.4 = 5.6 - .9y$.
28. $3.7d + 7 = 2.9d + 9.2 - 1.2d$.

118. Equations with Negative Roots. In solving equations involving negative roots, we shall likewise use only the principles that have been considered in the preceding chapters. This will become clear from a study of the following example:

Illustrative Example. Solve the equation

$$5x - 2 - 2x = x - 10.$$

Solution.

(1)	$5x - 2 - 2x = x - 10.$	Given.
(2)	$3x - 2 = x - 10.$	S. T.
(3)	$3x - x = -10 + 2.$	Tr.
(4)	$2x = -8.$	S. T.
(5)	$x = -4.$	D_2.

$n = 3$

Check. $5(-4) - 2 - 2(-4) = (-4) - 10.$

$$-20 - 2 + 8 = -4 - 10.$$

$$-14 = -14.$$

119. Changing Signs in an Equation. It is sometimes convenient to transform an equation by multiplying each member by -1. This process changes the sign of each term appearing in the equation.

Examine each of the following equations and observe how the solution is simplified by multiplying each member by -1.

	Original Form	New Form
(1)	$-4x = 8.$	$4x = -8.$
(2)	$2 - 3x = -10.$	$-2 + 3x = 10.$
(3)	$-5x + 6 = -2x.$	$5x - 6 = 2x.$
(4)	$-7 - 5x = 14 - 2x.$	$7 + 5x = -14 + 2x.$

Exercises. *Finding Negative Roots*

Solve each of the following equations and check each result:

1. $x + 7 = 5.$ -2
2. $12 = 18 + n.$
3. $2n + 17 = 9.$
4. $15 = 12 - p.$
5. $5d - 3 = -28.$
6. $x = 8x + 21.$

7. $3w - 10 = 5w$.

8. $12b = 7b - 25$.

9. $9y + 12 = y - 12$.

10. $7r - 6 = 3r - 30$.

11. $4a + 3 = 21 + 10a$.

12. $2t - 30 = 5t - 15$.

13. $s + 28 = 13 - 4s$.

14. $-4n + 9 = 2n + 12$.

15. $10 - 3d = d + 13$.

16. $6x - 3 + 2x = 9x + 5$.

17. $7g - 4 = 13g + 5 - 3g$.

18. $6s + 7 - s = 8s + 11$.

19. $4x - 3.2 + x = 2x - 10.4$.

20. $7 - 3d - 2.7 = d - 8.1 - 8d$.

21. The sum of two numbers is -39, and one number is 15 more than the other. Find the numbers.

22. The sum of three numbers is -8. The second is 5 more than the first, and the third is 1 less than twice the first. What are the numbers?

23. The length of a rectangle is 15 inches more than twice its width. If the perimeter is 24 inches, what are its dimensions? Is this result possible?

24. The total profit from a certain business, for 2 years, was \$2200. During the second year the profit was \$500 more than during the first year. What was the profit of each year? Interpret your answers.

25. A real-estate agent sold three houses. His profit from the second was \$1300 more than that from the first, and the profit from the third was \$400 less than twice what he made on the second. If his total profit was \$1500, how much did he make on each sale?

120. Equations involving Parentheses. When more than one unknown quantity appears in the same problem, it is often necessary to use parentheses in forming the equation that represents the conditions of the problem. It is important, therefore, to develop skill in solving such equations.

Illustrative Example. The sum of two numbers is 29. Four times the larger number exceeds five times the smaller number by 8. Find the numbers.

Solution. Let $n =$ the larger number.

Then $29 - n =$ the smaller number.

But 4 × (larger number) − 5 × (smaller number) = 8.

Hence we have the following equation and its solution:

(1) $4n - 5(29 - n) = 8.$ Given.

(2) $4n - 145 + 5n = 8.$ Why?

(3) $9n - 145 = 8.$ S. T.

(4) $9n = 153.$ $A_{145}.$

(5) $n = 17.$ $D_9.$

Hence (6) $29 - n = 12.$ Why?

That is, the required numbers are 17 and 12.

Check. $4 \times 17 - 5 \times 12 = 8.$

That is, $68 - 60 = 8,$

or $8 = 8.$

Exercises. *Equations containing Parentheses*

Solve each of the following equations and check each result:

1. $3(x + 2) = 12.$
2. $7(x - 5) = 21.$
3. $4(3x + 7) = 4.$
4. $5(2x - 9) = 15.$
5. $8(3a - 2) + 5 = 13.$
6. $12(a + 5) - 15 = -3.$
7. $9(5a - 8) - 20a = 3.$
8. $9n + 33 = 6(4n + 3).$
9. $7(a + 12) = 14 - 3a.$
10. $15(2n - 1) + 8 = 5n - 57.$
11. $2(4n + 11) = 3(5n - 2).$
12. $9x - 2(x + 5) = 11.$
13. $12 + 3(2x - 1) = 6.$
14. $5n + (4n + 6) = 42.$
15. $7y - (4y - 1) = 28.$
16. $9n - (3n + 5) = n - 50.$
17. $5(x - 2) = 4(2x - 4).$
18. $3(2s + 5) - 2(5s - 1) = 25.$

19. $3(d+9)=4(5-3d)+22$.

20. $8(3n+1)=21+(8n-5)$.

21. $10n-3=(6n-11)-(n+7)$.

22. $12a-(9-3a)=4(6a-9)$.

23. $3x+x(x-5)=8+(2+x^2)$.

24. $4y^2-2y(2y+7)=4(15-y)$.

25. $2(b^2-3)-b(5+2b)+26=0$.

26. $h(4h-1)-(3h-3)=4h^2$.

27. $x(3x-1)-(5x+3)=3(x^2-2)$.

28. $6(d-3)-5(4+3d)+10=2(3d+16)$.

29. $6n-2(4-n)+12=10-(4n+12)$.

30. $3(7w-1)-25=4(w+3)+(3w-5)-7$.

31. A line 20 inches long is to be divided into two parts such that one part is 8 inches more than the other. Find the two parts.

32. A board 16 feet long is to be sawed into two pieces one of which is to be 3 feet longer than the other. Find the length of each piece.

33. A rectangular building lot has a perimeter of 282 feet. Its length is 5 feet more than three times its width. What are its dimensions?

121. Equations with Fractional Coefficients. Many problems lead to equations which involve fractional coefficients. An equation of this sort may easily be transformed into one which contains only whole numbers, by multiplying both members of the equation by a number which contains each of the denominators as one of its factors. The resulting equation can then be solved in the usual way.

Illustrative Example 1. Solve the equation

$$\frac{x}{3}+2=\frac{1}{3}.$$

*Solution.** Multiplying both members by 3, we have

$$3\left(\frac{x}{3}+2\right)=3\times\frac{1}{3}.$$

By the Rule of Distribution, we have

$$x+6=1.$$

Hence, by Axiom 2, $x=-5.$

Check. Substituting -5 for x in the given equation, we have

$$\frac{-5}{3}+2=\frac{1}{3},$$

or $\frac{1}{3}=\frac{1}{3}.$

Illustrative Example 2. Solve the equation

$$\frac{3n}{4}-\frac{n}{8}=\frac{5}{2}.$$

Solution. (1) $8\left(\frac{3n}{4}-\frac{n}{8}\right)=8\times\frac{5}{2}.$ | *M_8.*

(2) $6n-n=20.$ | Why?

(3) $5n=20.$ | S. T.

(4) $n=4.$ | *D_5.*

Check. $$\frac{3\times 4}{4}-\frac{4}{8}=\frac{5}{2},$$

or $3-\frac{1}{2}=\frac{5}{2},$

or $\frac{5}{2}=\frac{5}{2}.$

In the exercises which follow, only simple fractions will be used. Equations of a more complicated type will be considered in a later chapter.

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* You will recall that when a fraction is multiplied by a whole number, its numerator only is multiplied by that number. Thus,

$$4\times\frac{3}{5}=\frac{12}{5},\quad 3\times\frac{5}{7}=\frac{15}{7},\quad \text{and so on.}$$

Similarly, in algebra, $3\times\frac{x}{5}=\frac{3x}{5}$, $5\times\frac{3x}{4}=\frac{15x}{4}$, and so on.

Exercises. *Equations with Fractional Coefficients*

Solve each of the following equations and check each solution:

1. $\frac{x}{2}+3=5.$

2. $\frac{x}{5}+\frac{x}{10}=3.$

3. $\frac{n}{6}+4=\frac{n}{3}.$

4. $\frac{d}{2}-\frac{d}{6}=\frac{2}{3}.$

5. $\frac{3y}{4}-5=y.$

6. $\frac{2b}{5}-\frac{b}{15}=\frac{5}{3}.$

7. $\frac{2n}{5}+\frac{7}{20}=\frac{3n}{4}.$

8. $\frac{5n}{3}-\frac{1}{4}=\frac{5n}{12}+1.$

9. $\frac{c}{2}-\frac{5c}{8}+\frac{3c}{4}=5.$

10. $a-\frac{a}{3}=\frac{12}{9}.$

11. $\frac{d}{3}+4=\frac{d}{6}.$

12. $\frac{y}{8}-\frac{2y}{3}=\frac{13}{24}.$

13. $x-\frac{x}{5}=20.$

14. $n+\frac{n}{20}+\frac{n}{4}=52.$

15. $\frac{b}{6}-4=\frac{5b}{18}.$

16. $\frac{3r}{4}=\frac{5r}{8}-4.$

17. $\frac{n+5}{12}=\frac{3}{4}.$

18. $\frac{s+4}{4}=\frac{s}{8}.$

19. $\frac{n}{3}+\frac{n}{5}+14=n.$

20. $\frac{t-3}{6}=\frac{t}{5}+2.$

21. $.3x+.2=2.3.$

SUGGESTION. Multiply both members of the equation by 10, which will remove the decimals.

22. $.25n=.13n+24.$

23. $.35x+2.1=.14x.$

24. $.15y=.25(y-2).$

25. $.75(n+20)=n-10.$

26. The difference between $\frac{3}{4}$ of a number and $\frac{1}{8}$ of the number is 35. Find the number.

27. The width of a building lot is $\frac{3}{8}$ of its length. Find its dimensions if the perimeter is 330 feet.

28. A farmer decided to sell $\frac{1}{3}$ of his apple crop, to keep $\frac{1}{4}$ of it, and to put the remainder in cold storage. If he put 50 bushels in cold storage, how many bushels were there in his entire crop?

29. A coat which had been marked down $\frac{1}{5}$ sold for \$24.60. What was the original price?

30. There are two numbers such that the larger is 6 more than the smaller. One half of the larger is 7 more than one sixth of the smaller. Find the numbers.

122. Literal Equations. A *literal equation* is one in which at least one of the coefficients or terms appearing in the equation is represented by a letter, aside from the letter representing the unknown quantity. For example, $x+a=b$, $3x+c=d-5$, and $x-3=p$ are literal equations in which x represents the unknown quantity. Such an equation may be solved by applying the rules stated previously.

Sometimes it becomes necessary to find one of the letters of a *formula* in terms of the other letters. Simple illustrations of such transformations were suggested in a previous chapter. If a formula is more complicated, it is best to regard it as a literal equation. The four fundamental axioms or the special methods explained previously may then be used to express any letter in terms of the other letters, as shown in the following examples:

USING ADDITION	USING SUBTRACTION	USING MULTIPLICATION	USING DIVISION
$x-b=c$	$x+b=c$	$\frac{x}{a}=b,$	$ax=b,$
$\underline{\quad b=b}$	$\underline{\quad b=b}$	$a\cdot\frac{x}{a}=a\cdot b,$	$\frac{ax}{a}=\frac{b}{a},$
$x=c+b$	$x=c-b$	$x=ab.$	$x=\frac{b}{a}.$

Illustrative Example 1. If $2x - 4b = 10c + 2b$, find x.

Solution. (1) $2x = 10c + 6b$. | Tr. and S. T.
(2) $x = 5c + 3b$. | D_2.

Illustrative Example 2. If $ab + c = d$, find a.

Solution. (1) $ab = d - c$. | S_c.
(2) $a = \frac{d-c}{b}$. | D_b.

Exercises. *Literal Equations*

Numbers 1 to 14, oral

Solve the following equations for x or y:

1. $2x = n$.
2. $nx = 4$.
3. $ay = b$.
4. $x + 3 = n$.
5. $y + n = 8$.
6. $y + d = c$.
7. $x - 5 = a$.
8. $x - n = 4$.
9. $y - r = s$.
10. $2x + 3n = 9n$.
11. $3y - a = b$.
12. $\frac{x}{3} = c$.
13. $\frac{y}{d} = 6$.
14. $\frac{x}{n} = a$.

Solve the following equations for x or y, and check each result:

15. $2x + 6 = 4n$.
16. $5y - 6a^2 = 4a^2$.
17. $3x - 12c = 3d$.
18. $4y + 4x = 7b$
19. $2y - 5d = 10d - 3y$.
20. $c + 3y = 2y + d$.
21. $7x - 3m = 4x + 6n$.
22. $3(x - a) = x + 7a$.
23. $8s - 2(y + 3s) = 12s$.
24. $2(5y - 3a) = 4(y - 3b)$.
25. $ax - ab = a^2$.
26. $nx + c = 2r$.
27. $3b^2y = 2b(6b^2 + by)$.
28. $4c(5c - 2x) = 3(4c^2 - 2cx)$.

29. $\frac{y}{2} - \frac{h}{3} = \frac{h}{6}$.

30. $\frac{x}{a} + a = 3$.

31. $\frac{x-b}{3} = 2d$.

32. $2 + \frac{2x}{10r} = \frac{x}{3r}$.

The formulas below are literal equations which may be solved for any one of the letters used in terms of the remaining letters.

33. Solve $A = lw$ for w; for l.

34. Solve $P = W + F$ for W; for F.

35. Solve $p = s - c$ for s; for c.

36. Solve $D = \frac{M}{V}$ for M; for V.

37. Solve $A = \frac{bh}{2}$ for h; for b.

38. Solve $v = u + at$ for u; for t; for a.

39. Solve $A = \frac{u+v}{2}$ for u; for v.

40. Solve $F = \frac{Wa}{g}$ for a; for g; for W.

41. Solve $w = \frac{2bd^2}{3l}$ for b; for l.

42. Solve $l = a + d(n-1)$ for a; for n.

43. Solve $A = p + prt$ for t; for r.

44. Solve $S = 2\pi rh$ for h; for r.

45. Solve $T = \frac{l}{a} + t$ for l; for t.

123. Percentage Problems. By using the formulas $p = br$ and $i = prt$, as well as the short methods of solving equations which have been considered in this chapter, you will now be able to solve more difficult applied problems involving percentage. In the following exercises use the suggestions which have been given previously for solving such problems. In the case of interest problems, only simple interest is to be computed.

Exercises. *Percentage Problems*

1. An agent sells a house for \$9350, making a profit of 10 per cent. What was the original value of the house?

2. A school building cost \$367,500, which exceeded the original estimate by 5 per cent. What was the estimated cost?

3. Two equal sums of money are invested, one at 4 per cent and the other at 5 per cent. If the total interest from both investments is \$270 a year, how large is each sum?

4. Two sums of money are invested at 4 per cent and 6 per cent respectively. The second is \$2000 larger than the first. If the combined annual income from both investments is \$360, what are the two sums?

SUGGESTION. The conditions of the problem may be tabulated as follows:

	PRINCIPAL	RATE PER CENT	INTEREST
First sum . .	p	4%	$.04\,p$
Second sum. .	$p + 2000$	6%	$.06(p + 2000)$

Hence the required equation is

$$.04\,p + .06(p + 2000) = 360.$$

5. Two sums of money, the first exceeding the second by \$1500, were invested at 4 per cent and 5 per cent respectively. The larger sum yielded \$15 more each year than the other. How large were the two sums?

6. How many dollars must be invested at 4 per cent to yield an annual income of $3000?

7. A man invests $4000, a part at 4 per cent and the remainder at 6 per cent. If the 4 per cent investment yields annually $60 more than the 6 per cent investment, how many dollars were invested at each rate?

8. A man invests $2000 at 4 per cent, $3000 at 5 per cent, and $1200 at 6 per cent. What rate of interest is he receiving from the three investments combined?

9. At what rate is a sum of money invested if the interest each year equals one tenth of the principal? one twentieth of the principal?

SUGGESTION. In this case $\frac{r}{100} \cdot p = \frac{1}{10} \cdot p$.

Dividing by p, we have $\frac{r}{100} = \frac{1}{10}$.

Hence $r = 10$.

That is, the rate is 10%.

10. In how many years will the interest on a principal invested at 4 per cent simple interest equal the principal?

SUGGESTION. Since $\frac{4}{100} \cdot p \cdot t = p$,

we have $\frac{4}{100} t = 1$;

whence $t = 25$.

11. What is the result, in Exercise 10, if the rate is 5 per cent? $4\frac{1}{2}$ per cent? $3\frac{1}{2}$ per cent?

12. In how many years does a sum invested at 5 per cent simple interest become three times as large?

13. In measuring a line which was 25 inches long a pupil made a mistake of $\frac{1}{4}$ inch. What was the relative error? the percentage error?

EXPLANATION. All measurements are only approximately correct, since they always involve an error. By dividing the *actual error* of a measurement by the numerical value under consideration, we obtain the *relative error*. If the relative error is expressed in per cent form, we obtain the *percentage error*. For example, suppose that in measuring a line which is 25 inches long you make an error of $\frac{1}{2}$ inch. Then the *relative error* is $\frac{\frac{1}{2}}{25}$, or $\frac{1}{50}$, and the *percentage error* is 2%.

14. In measuring a line that was 4 feet long a pupil made a mistake of 5 per cent. What was the actual error, in inches?

15. The length of a playground was 250 feet. It was measured by a pupil, with a percentage error of $\frac{1}{2}$ of 1 per cent. What was the actual error, in inches?

16. A certain business involved an investment of $10,000. During the first year there was no profit. The second year showed a gain of 3 per cent. During the third year there was a profit of 2 per cent. After that the annual profit was 9 per cent. After 8 years what was the average rate of interest which this investment had yielded?

17. Suppose that on a debt of $5000 the interest rate is increased each year by 1 per cent. If after 3 years this debt amounts to $5750, what is the rate of interest for each year?

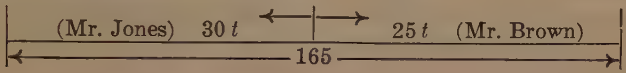
124. Motion Problems. The formula $d = rt$, which was introduced in a previous chapter, enables us to solve many interesting problems that involve a uniform rate of motion.

Illustrative Example 1. Two neighbors, Mr. Brown and Mr. Jones, start on an automobile trip at the same time, one going east and the other going west. Mr. Brown travels at the rate of 25 miles per hour, and Mr. Jones at the rate of 30 miles per hour. After how many hours will they be 165 miles apart?

Solution. Let t represent the number of hours. Then the conditions of the problem may be tabulated as follows:

	RATE (r)	TIME (t)	DISTANCE (d)
Mr. Brown	25	t	$25\,t$
Mr. Jones	30	t	$30\,t$

A diagram such as the following may also be used to picture the conditions of the problem:



Hence, since the sum of the two distances equals 165, we have

$$25\,t + 30\,t = 165.$$

Solving the equation, we find that $t = 3$.

That is, in 3 hours the two men will be 165 miles apart.

Check. Show that $3 \times 25 + 3 \times 30 = 165$.

Illustrative Example 2. Henry started from camp one morning at 6 A.M. to get some supplies in the nearest town. He walked at the rate of 3 miles an hour. At 8 A.M. his friend Jack followed him on his bicycle, going at the rate of 9 miles an hour. At what time, and how far from camp, did he overtake Henry?

Solution. The conditions of the problem may be tabulated as follows:

	RATE	TIME	DISTANCE
Henry	3	t	$3\,t$
Jack	9	$t - 2$	$9(t - 2)$

Since Jack overtakes Henry, the two distances are equal.

Hence $$9(t - 2) = 3\,t.$$

Solving this equation, we find that

$$t = 3.$$

(Can you complete the solution?)

Exercises. *Motion Problems*

1. Two trains start from the same station at the same time, one going north and the other going south. Their hourly rates are 40 and 50 miles respectively. After how many hours will the trains be 360 miles apart?

2. Two trains start from the same place at the same time, traveling in opposite directions. One travels twice as fast as the other. If they are 360 miles apart in 4 hours, find the speed of each train.

3. A starts from a certain place and walks at the rate of 4 miles per hour. Two hours later B starts from the same place and travels in the same direction at the rate of 6 miles per hour. In how many hours will B overtake A?

4. In Exercise 3, how many hours after B started will his distance exceed that of A by 10 miles?

5. In Exercise 3, how many hours after A started will he be only 2 miles ahead of B?

6. Two trains start east over the same road, at the same time, one from Detroit and the other from a point 30 miles west of Detroit. If the first train runs at a rate of 40 miles per hour and the second at a rate of 50 miles per hour, in how many hours will the second train overtake the first?

7. In the preceding problem, in how many hours will the faster train be only 5 miles behind the other?

8. A messenger is sent from a camp at 7 A.M. to the nearest post office. Another messenger follows him at 8 A.M. and overtakes him after 1 hour and 30 minutes. If the second messenger travels 4 miles an hour faster than the first, what is the rate of each?

9. A freight boat which is averaging 10 miles per hour is 900 miles out at sea when it is followed by a fast passenger boat. If the passenger boat makes 25 miles an hour, in how many days will it overtake the freight boat?

10. A passenger train averaging 38 miles an hour leaves one terminal station of a railroad at 5 P.M. At the same time a freight train running at a rate of 22 miles per hour leaves the other terminal station. If the two stations are 380 miles apart, at what time will the trains pass each other?

11. A sailboat starts out from a harbor, going at the rate of 8 miles per hour. It is followed after 2 hours by a motor boat which is going at the rate of 16 miles per hour. How long will it take the motor boat to overtake the sailboat?

12. In an automobile race one car makes 60 miles per hour. A second goes at an average speed of 54 miles per hour. If the second car has started 20 minutes before the other car, how soon will it be overtaken by the faster car?

13. A train which averages 32 miles per hour leaves Chicago for Washington at 11 A.M. It is followed on the same road at 2 P.M. by a fast passenger train. In how many hours will the fast train overtake the first train if the fast train averages 50 miles per hour?

14. A steamship which is in distress sends out an SOS signal by wireless. The ship is then 390 miles from the nearest harbor and is able to make only 6 miles per hour. How soon will a relief ship sent from the harbor at the rate of 24 miles per hour be able to reach the disabled ship?

15. The distance from New York to Liverpool, on the route ordinarily chosen, is about 3105 miles. A passenger

boat leaves New York for Liverpool at 9 A.M. on Saturday, and averages 22 miles per hour. A second boat leaves Liverpool for New York at 9 A.M. on the following Tuesday, Eastern Standard Time. When will the boats meet if the second boat makes 23 miles per hour?

125. Mixture Problems. In the grocery business it is often necessary to mix various qualities of groceries in order to obtain a "blend" that will sell at a reasonable price. In like manner, "mixture problems" must be solved by druggists, jewelers, chemists, manufacturers, and so on. A few of the simplest types of these problems will be introduced in the following exercises.

Illustrative Example. A wholesale grocery firm wishes to mix two grades of coffee, worth 40 and 60 cents a pound, so as to obtain a grade that can be sold at 42 cents a pound. How many pounds of each kind must be used to obtain 90 pounds of the mixture?

Solution. Let $n =$ the number of pounds of 40-cent coffee.
Then $90 - n =$ the number of pounds of 60-cent coffee.
Hence $40n =$ the value of the 40-cent coffee, in cents,
and $60(90 - n) =$ the value of the 60-cent coffee, in cents.
But $90 \times 42 =$ the value of the mixture, in cents.
Hence $40n + 60(90 - n) = 3780.$

Solving this equation, we find that

$$n = 81.$$

(Can you complete the solution?)

Exercises. *Mixture Problems*

1. A wholesale house wishes to blend 200 pounds of 55-cent coffee with 300 pounds of a cheaper grade, and thus be able to sell the new blend for 46 cents a pound. What grade of coffee must be used for blending?

2. A grocer decides to mix almonds which sell for 25 cents a pound with 120 pounds of brazil nuts which sell for 38 cents a pound. How many pounds of almonds must he use if he intends to sell the mixed nuts for 30 cents a pound?

3. A confectioner wishes to mix candy which sells for 30 cents a pound with a grade that sells for 50 cents a pound, so that he can sell the mixture for 38 cents a pound. If he needs 50 pounds of this mixture, how many pounds of each kind should he use?

4. A grocer needs 30 pounds of mixed spices which are to sell at 35 cents a pound. How many pounds of spices which sell for 25 cents a pound and for 40 cents a pound, respectively, must he use to get the desired mixture?

5. How much alcohol must be added to 8 ounces of a 25 per cent solution of iodine to reduce it to a 10 per cent solution?

SUGGESTION. If n ounces of alcohol are added, the mixture will contain $(n + 8)$ ounces. Hence $\frac{10}{100}(n + 8) = \frac{25}{100} \times 8$. Multiply each member by 100.

6. How much water must be added to 3 ounces of a 15 per cent solution of argyrol to reduce it to a 10 per cent solution?

7. Camphor gum is dissolved in alcohol to make spirits of camphor. How many ounces of alcohol must be added to 24 ounces of a 15 per cent camphor solution to make a 10 per cent solution?

8. How much pure silver must be added to 120 ounces of an alloy containing 75 per cent of silver, so that the resulting alloy shall be 90 per cent pure?

SUGGESTION. If n ounces of silver are added, we have

$$\frac{90}{100}(n + 120) = \frac{75}{100} \times 120 + n. \quad \text{(Why?)}$$

9. An auto radiator contains 8 quarts of a 10 per cent solution of alcohol. How much alcohol must be added to make it a 30 per cent solution?

10. A school ordered 300 pounds of nitric acid 67 per cent pure. How much water had to be added to reduce it for use in the laboratory, which required only a 30 per cent solution?

11. Sulphuric acid 98 per cent pure is purchased by a school for use in its chemical laboratory. How much water must be added to 150 pounds to reduce it to a 15 per cent solution, the strength desired for laboratory use?

12. A firm ordered 3675 pounds of sulphuric acid 80 per cent pure. The manufacturer shipped a smaller quantity of a solution 98 per cent pure, in order to save on transportation. How much water had to be added to the shipment to obtain the desired quantity of an 80 per cent solution?

13. How much water must be evaporated from 30 gallons of a 15 per cent salt solution so that the remainder will be a 25 per cent solution?

14. How many quarts of cream containing 25 per cent butter fat and of milk containing 5 per cent butter fat must be mixed so as to obtain 30 gallons of cream which will contain 20 per cent butter fat?

Exercises. *Miscellaneous Problems*

1. When 5 is subtracted from one half of a number, the result is 6. Find the number.

2. The sum of one half of a number and one third of the number is 10. Find the number.

3. If one fifth of a number is subtracted from 55, the result is twice the number. Find the number.

4. Louise has 54 cents, and Helen has 32 cents. How many cents does Louise have to give to Helen so that each may have the same number of cents?

5. James has a dollars, while Robert has only b dollars. How many dollars does James have to give to Robert so that each may have the same number of dollars?

6. A father divides $63 among his three sons and two daughters, giving to each son $1 more than to a daughter. How many dollars does he give to each son and daughter?

7. In the assembly hall of a school it was found that when 30 pupils were seated in a row, only one seat was occupied in the last row. If only 29 pupils had been seated in a row, one pupil would have been without a seat. How many rows were there, and how many pupils?

AGE PROBLEMS

8. Paul is one third as old as his father. Five years ago his father was four times as old as Paul. How old is each now?

9. Three years ago John was half as old as he will be 6 years from now. What is his present age?

10. Mary's mother said, "If you were 2 years older, you would be one third as old as Uncle Henry, who is now 36 years old." What is Mary's age?

11. William is three fourths as old as his brother Fred, but 3 years from now he will be five sixths as old as Fred. How old is each?

12. A father's age is 36, and that of his son is 3 years. In how many years will the father be four times as old as his son?

COIN PROBLEMS

13. James had \$1.65 in dimes and nickels. If there were 18 coins in all, how many of each kind did he have?

SUGGESTION. $10n + 5(18 - n) = 165$. (Explain.)

14. Henry changed a one-dollar bill in the grocery store, receiving in return 14 coins made up of dimes and nickels. How many of each kind did he receive?

15. I changed a five-dollar bill and received in return half dollars, quarters, and dimes. I received as many half dollars as quarters. If there were 17 coins in all, how many of each kind did I receive?

16. During a certain month my expense for postage was \$1.40. I sent only letters at 2 cents each and postcards at 1 cent each. How many of each kind did I send if I mailed in all 90 letters and postcards?

GEOMETRIC PROBLEMS

17. A line which is 12 inches long is to be divided into three parts such that each is 1 inch longer than the preceding part. Find the length of each part.

18. A line 70 feet long is to be divided into five parts such that each part is 2 feet longer than the preceding part. Find the length of each part.

19. The length of a rectangle is twice the width. If each dimension were increased by 5 feet, the perimeter would be 80 feet. Find the dimensions.

20. The side of a square is x. If the side were $x + 1$, the area would be 17 square units larger. Find x.

21. The altitude of a triangle exceeds the base by 4 inches. If the altitude is increased by 2 inches, the area is increased by 10 square inches. Find the base and the altitude.

22. Two angles of a triangle are equal. The third angle is twice as large as either of the other two. How many degrees are there in each of the three angles?

Explanation. It can be proved that the sum of the angles of a triangle is 180 degrees.

23. One angle of a triangle is 87 degrees. The second angle is half as large as the third. How many degrees are there in each angle of the triangle?

24. If the sizes of the three angles of a triangle, measured in degrees, are denoted by consecutive integers, how many degrees are there in each angle of the triangle?

126. Summary. In this chapter you have learned

(1) How to solve an equation by interchanging members.
(2) How to transform an equation by addition or subtraction.
(3) How to solve equations with negative roots.
(4) How to transform an equation by changing signs.
(5) How to solve an equation involving parentheses.
(6) How to solve equations with fractional coefficients.
(7) How to solve literal equations.
(8) How to solve percentage problems, motion problems, mixture problems, and other applied problems.

Test on Chapter IX

The equations appearing in the second column below were obtained by changing the corresponding equations in the first column by one of these three methods: (I) *multiplying by* -1; (II) *interchanging members;* (III) *transforming by addition or subtraction. Indicate the method used in each case by writing I, II, or III.*

1. $7n - 4 = 8 + 5n$. $\qquad$ $8 + 5n = 7n - 4$.
2. $7 - 2x = x + 8$. $\qquad$ $-2x - x = 8 - 7$.
3. $6a - 10 = 2 + 3a$. $\qquad$ $-6a + 10 = -2 - 3a$.

Transform the equation $8y - 3 = 12y + 21 - 2y$, *as follows:*

4. Using Method I.
5. Using Method II.
6. Using Method III,

Solve each of the following equations:

7. $4x - (x + 12) = 2(21 - 3x)$.
8. $\frac{n}{3} + \frac{5n}{6} = \frac{7}{2}$.
9. $-4x - 15 = 9$.
10. $-7a + 4 = -3a - 8$.
11. $\frac{w+1}{2} = w - 12$.
12. $5b - y = 8b + 3 - b$.
13. In the formula $P = 2(l + w)$, solve for w.

Below each of the following problems you will find three equations. Select the one which correctly represents the conditions of the problem.

14. The length of a picture frame is 8 inches less than three times its width. If the perimeter is 32 inches, what are its dimensions?

(1) $2(4w - 8) = 32$. (2) $4w - 8 = 32$. (3) $2(3w - 8) = 32$.

15. A man sold his car for $576, thereby losing 40 per cent of the original cost. How much did he pay for the car originally?

(1) $x - 40 = 576$. (2) $x - \frac{2x}{5} = 576$. (3) $x + .40 = 576$.

Exercises. *Review*

1. Given that $A = 6n^2 + 7n - 5$, $B = n^2 - 5n - 3$, and $C = 2n - 1$, find

(1) $A + B + C$;
(2) $C - B$;
(3) $B \times C$;
(4) $A \div C$.

Check the result obtained in each case.

2. Remove the parentheses and combine the similar terms in each of the following:

(1) $3a(a^2 - 5a - 8)$. (3) $10 - 7(1 + 4n - s) + 3n$.
(2) $7x - 2(x - 3)$. (4) $5n(2n - 3) - (12 + 7n^2)$.

3. Multiply $-6xy^3$ in succession by 5, $3x^3$, $-2xy$.

4. Divide $36a^3b^5c$ in succession by -9, $+3c$, $-4ab^2$.

5. In a certain city the monthly cost of electricity may be found by the formula $c = 1.25 + .04e$, in which e represents the amount of electricity used and c the cost in dollars. Find the value of c when $e = 60$.

6. The formula for the tension in a cable is $T = 3 + \frac{wa}{g}$.

(1) Find T when $w = 200$, $a = 12$, and $g = 32$.
(2) Find a when $T = 1875$, $w = 1200$, and $g = 32$.

7. The average amount of life insurance sold in the United States per day, in millions of dollars, during six successive years, beginning in 1922, was as follows: 17, 21, 22, 25, 26, 27.

(1) Draw a graph representing these statistical facts.

(2) Indicate by a line the average daily sales for the six-year period.

(3) Indicate by means of signed numbers how the figure given for each year varies from the average.

8. The cost of sending a telegram to a certain city, for messages of various lengths, is shown in the following table:

Number of words	10	14	20	24	30	40
Cost in cents	42	52	67	77	92	117

Find a formula for the cost of a telegram of ten or more words. (Use c and w.)

CHAPTER X

EQUATIONS OF THE FIRST DEGREE IN TWO UNKNOWNS

1. GRAPHS OF EQUATIONS

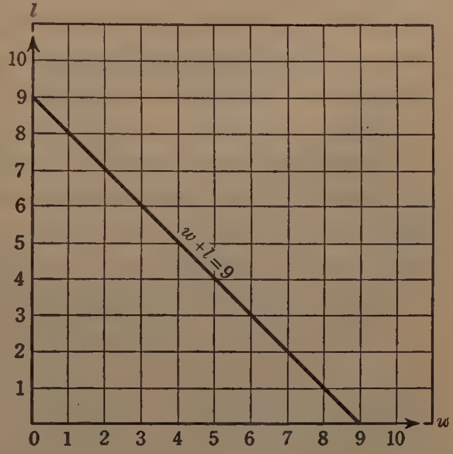
127. Picturing an Equation. In Chapter VI you learned how to draw graphs of simple formulas such as $c = 0.20\,g$. The two letters of this formula enabled us to make a table containing two related sets of numbers. By using two axes, it was shown how we may obtain a picture, or a graph, of such a table. We shall now extend this work by considering the graphs of *equations* having two unknown quantities.

Illustrative Example. In making a window frame, a carpenter used two boards each 9 feet long. He decided to cut each into two pieces, one representing the length and the other representing the width. What dimensions might he select for the window?

Solution. If w denotes the width and l denotes the length, the problem is represented by the equation $w + l = 9$. This equation calls for *two* unknown values. Obviously, there are many answers. Thus, if $w = 3$, we find that $l = 6$; if $w = 4$, then $l = 5$; and so on. Many *pairs* of *values* might thus be selected which satisfy this equation. This fact is shown more clearly by the following table:

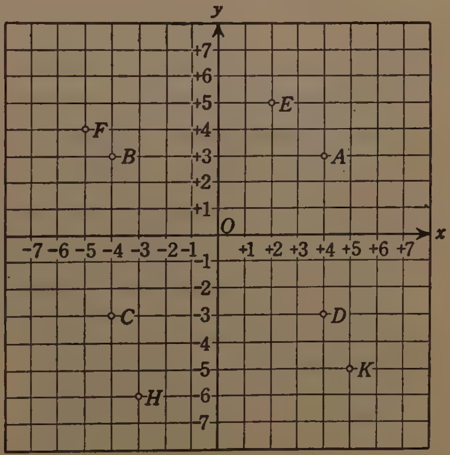
If $w =$	0	1	2	3	4	5	6	7	8	9
then $l =$	9	8	7	6	5	4	3	2	1	0

How such a table may be represented graphically is shown in the accompanying diagram. Observe that the points which we located by means of the table appear to lie on a *straight line.*



From this graph we may find, for *any* value of *w*, in the problem suggested on page 260, the corresponding value of *l*. Moreover, from *any point* in the graph we may obtain a pair of values which satisfies the given equation. Hence the graph is a complete pictorial solution of the problem which the equation represents.

128. Locating Points. To picture a table such as the one given on page 260, we must use two *axes.* The horizontal axis is usually called the *x-axis,* and the vertical axis is called the *y-axis.* The point *O,* where the axes intersect, is called the *origin.*



If these axes are drawn on a sheet of squared paper, evidently we can locate any point on the paper if we know its distance and direction from each of the axes. Now, a point may be either to the *right* or to the *left* of the *y*-axis, and either *above* or *below* the x-axis. Hence it will be easier to locate a point if on each axis we mark off a scale of positive and negative distances. Distances to the *right* of the y-axis or *above* the x-axis are regarded as *positive*, whereas distances to the *left* of the y-axis or *below* the x-axis are regarded as *negative.*

This agreement makes it possible to locate a point by means of two signed numbers. Thus, point A, in the diagram on page 261, is 4 units to the *right* of the vertical axis and 3 units *above* the horizontal axis. Hence its "x number" is $+4$, and its "y number" is $+3$.

The two signed numbers which help us to locate a point are called the *coördinates* of the point. The *x number* is called the *abscissa* of the point, and the *y number* is the *ordinate* of the point. In giving the coördinates of a point, it is customary to give the *abscissa first.* Thus, the abscissa of point A is $+4$, and its ordinate is $+3$. This is indicated briefly by writing $(+4, +3)$. Similarly, the location of point B is indicated by writing $(-4, +3)$, while the coördinates of C and D are $(-4, -3)$ and $(+4, -3)$ respectively.

Locating a point by the method which has just been described is often called *plotting* that point.

Exercises. *Locating Points*

1. In the diagram on page 261 what are the coördinates of the points E, F, H, K?

2. On a sheet of squared paper draw an x-axis and a y-axis. Mark off on each a positive and a negative number scale, and then locate the following points:

(1) $(4, 5)$.
(2) $(-3, 2)$.
(3) $(-5, -1)$.
(4) $(-4, 6)$.
(5) $(0, 5)$.
(6) $(-6, 0)$.
(7) $(0, 7)$.
(8) $(-7, 5)$.

3. Locate the points A, B, and C which have coördinates as follows: A, $(3, 4)$; B, $(-2, 4)$; C, $(1, -2)$. Connect the points A, B, and C, thus forming the triangle ABC.

4. (1) Plot the following points: A, $(3, 2)$; B, $(-3, 2)$; C, $(-3, -2)$; D, $(3, -2)$. Connect these points in succession. What kind of figure is $ABCD$? (2) Draw the diagonals AC and BD. They intersect at a point P. What are the coördinates of P?

5. Locate the following points: $(-4, -4)$, $(-3, -3)$, $(-2, -2)$, $(-1, -1)$, $(0, 0)$, $(1, 1)$, $(2, 2)$, $(3, 3)$, $(4, 4)$. Connect these points in succession. Does the graph appear to be a broken line or a straight line?

6. Plot the following points: $(-3, -6)$, $(-2, -4)$, $(-1, -2)$, $(0, 0)$, $(1, 2)$, $(2, 4)$, $(3, 6)$. Connect the points in succession. Does the graph appear to be a broken line or a straight line?

129. Graphs of Linear Equations. In the illustrative example on page 260 we obtained for l and w pairs of values that were seen to satisfy the equation $w + l = 9$. When these values were plotted, we obtained points that appeared to lie on a *straight line.*

Only *positive* values were considered in the example mentioned above. There are problems, however, which involve also *negative* values of the unknown quantities. How to deal graphically with simple cases of this kind will become clear from the following example:

Illustrative Example. I am thinking of two numbers such that their sum is 5. Find the numbers.

Solution. It is evident that there are many pairs of values that satisfy the conditions of the problem. Thus, if we think only of positive integers, we may have

$$1 + 4 = 5,\quad 2 + 3 = 5,\quad 4 + 1 = 5,\quad 0 + 5 = 5,\quad \text{and so on.}$$

If we are thinking of fractions and decimals, we may have

$$\tfrac{1}{2} + 4\tfrac{1}{2} = 5, \quad 2\tfrac{1}{3} + 2\tfrac{2}{3} = 5, \quad 1.9 + 3.1 = 5, \quad \text{and so on.}$$

If we allow also negative values, we have many more solutions such as

$$-1 + 6 = 5, \quad -3 + 8 = 5, \quad -5 + 10 = 5, \quad \text{and so on.}$$

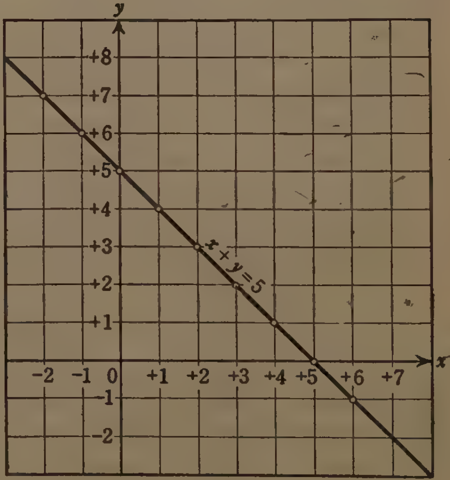
It is customary to represent any possible value of the first number by x, and any possible value of the second number by y. In that case the problem may be represented by the equation

$$x + y = 5.$$

Corresponding values of the two numbers may then be arranged in a table such as the following:

If $x =$	0	1	2	3	4	5	6	-1	-2	-3
then $y =$	5	4	3	2	1	0	-1	6	7	8

Now, each pair of corresponding numbers in the table, such as (1, 4), may be represented graphically by the method explained previously, as shown in the adjoining figure. If the points plotted from these pairs of values are connected in succession, it appears that they all lie on a straight line. This is a fact which can be proved, although the proof is too difficult for consideration at this point.



The straight line shown in the figure is said to be *the graph of the equation* $x + y = 5$. That is the reason why an equation of this kind is called a *linear equation.*

This graph is the complete pictorial solution of the problem stated above, since any point on the graph has for its coördinates values of x and y which satisfy the equation $x + y = 5$.

NOTE. The method of plotting points by means of coördinates and of representing an equation by a graph, which has been described in the preceding pages, is due chiefly to the genius of the great French mathematician and philosopher, René Descartes (1596–1650). He became interested in mathematics at an early age and soon made discoveries and contributions in several fields of work that cause his name to rank among the highest in the whole history of science and mathematics.

130. Definitions. 1. It is evident that an equation such as $x + y = 5$ is satisfied by many pairs of corresponding values of x and y. Whenever the value of x changes, or *varies*, the corresponding value of y also varies. For these reasons the two unknown quantities in such an equation are commonly called *variables* (or changing numbers).

These variables are definitely *related*. As you draw the graph of the equation pictured on page 264, the coördinates of its points constantly change, but always in such a way that $x + y = 5$.

2. An equation is said to be of the *first degree* in two unknowns such as x and y if (1) only one unknown appears in any term, (2) only the first power of either unknown is involved, (3) no unknown appears in any denominator.

For example, $4x - 5y = 12$ is of the first degree, while $7x - 3xy = 4$ and $3x - 3y^2 = 14$ are not of the first degree in x and y.

3. It can be shown that every equation of the first degree in x and y has for its graph a straight line. Hence such an equation is usually called a *linear equation.*

4. A single equation in two or more unknowns is called an *indeterminate equation.*

Thus, $x + y = 7$ is *indeterminate,* since many pairs of values satisfy the equation.

There are many indeterminate equations which are *not* linear. For example, the equation $y = 4x^2 + 7x + 6$ is indeterminate, though its graph is a curved line.

Exercises. *Graphs of Linear Equations*

1. Draw the graph of the equation $3x - 2y = 6$.

Solution. Two different methods may be used to find a set of corresponding values of x and y.

a. Direct Substitution. Assign any convenient values to x in the given equation and then find the corresponding values of y.

Thus, when $x = 0$, we have

$$0 - 2y = 6,$$

or

$$-2y = 6.$$

That is, $y = -3.$

Similarly, when $x = 4$, we have

$$12 - 2y = 6.$$

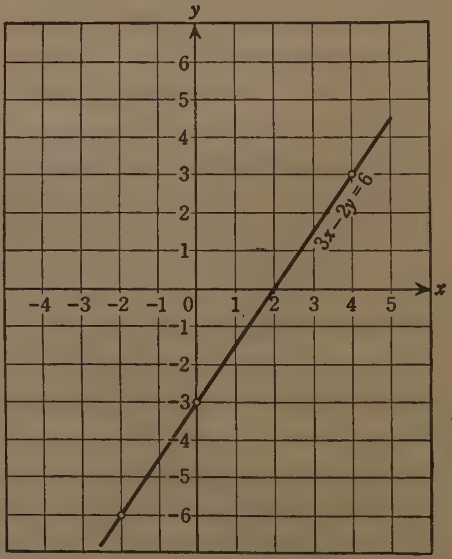
Hence $y = 3.$

Also, when $x = -2$, we have $-6 - 2y = 6.$

Hence $y = -6.$

This work may be done *mentally*. You will find it convenient to use the following tabular form:

If $x =$	-2	0	4
then $y =$	-6	-3	3



These pairs of values enable us to draw the graph of the given equation, as shown in the figure.

b. Transforming the Equation before Substitution. From the given equation find y in terms of x and then proceed as above.

Thus, if $3x - 2y = 6,$

we have $2y = 3x - 6.$

Hence $y = \frac{3x - 6}{2}.$

It is then easy to find values of x such that each corresponding value of y becomes a whole number.

Note. *Only two points are necessary to draw the graph of a linear equation.* For purposes of checking it is advisable, however, to plot *three* points which are conveniently located. Two pairs of values can often be obtained very easily by first making $x = 0$ and then making $y = 0$. Thus, in the equation above, when $x = 0$, $y = -3$; and when $y = 0$, $x = 2$.

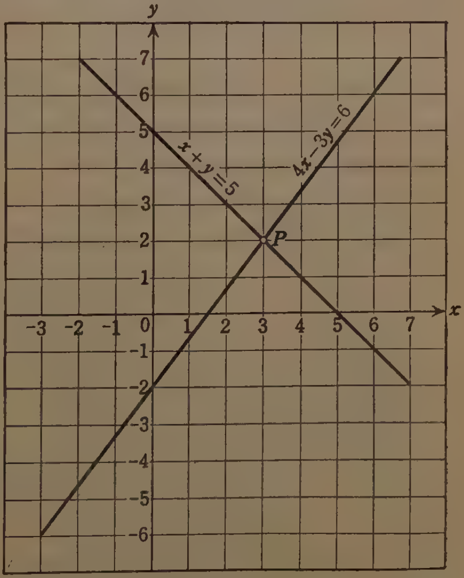
Draw the graph of each of the following equations:

2. $x + y = 4$.
3. $x - y = 5$.
4. $-x + y = 6$.
5. $x - 2y = 8$.
6. $2x - 5y = 10$.
7. $4x + y = 0$.
8. $5x - 3y = 6$.
9. $3x - 5y = 10$.
10. $7x - 2y = 4$.
11. $y = x$.
12. $y = x + 4$.
13. $y = 2x$.
14. $y = 4x - 2$.
15. $y = -x$.
16. $y = -2x - 3$.

17. Each of the following points lies on the graph of one of the equations in Exercises 11–16 above. In each case select the correct equation:

(1) (4, 4).
(2) (3, 7).
(3) (3, 10).
(4) (4, − 4).
(5) (4, 8).
(6) (2, − 7).

Note. Instead of x and y, we may use other letters to represent the two unknowns of a linear equation. For example, we may have $7a + 5b = 9$, $c = 7p - 2$, $d = 5r - 3$, and so on.



131. Simultaneous Equations. This figure shows the graphs of the linear equations $x + y = 5$ and $4x - 3y = 6$.

These graphs intersect at the point P. The coördinates of P are $x = 3$ and $y = 2$. It is evident that these values of x and y satisfy both equations. Moreover, since the two lines have no other point in common, (3, 2) is the only pair of numbers which will satisfy both equations.

Two equations which are satisfied by the same pair of values are called *simultaneous equations.*

Not all simultaneous equations are linear. Thus, $y = x^2 + 3$ and $3x + y = 7$ are simultaneous, but the first equation is not linear. Only pairs of *linear* equations will be considered in this course.

Since two straight lines can intersect in but one point, two simultaneous linear equations in two variables can have but one common solution. *To solve a pair of linear equations is to find that common solution.*

Exercises. *Solving Pairs of Equations by Graphs*

Solve graphically the following pairs of equations, and verify each solution by substitution.

1. $x + y = 4,$
   $x - y = 2.$
2. $x + y = 7,$
   $x - y = 5.$
3. $2x + y = 8,$
   $4x - y = 4.$
4. $2x - y = 2,$
   $x + 2y = 11.$
5. $x - 2y = 1,$
   $x + 2y = 9.$
6. $x - y = -1,$
   $2x + y = 10.$
7. $2x - y = -1,$
   $4x - y = 3.$
8. $3x + y = 0,$
   $4x + 3y = 5.$
9. $2x + 3y = 6,$
   $5x + 4y = 1.$
10. $x + y = 2,$
    $3x + 2y = -2.$
11. $2x + 3y = 2,$
    $2x + 6y = 3.$
12. $4x + y = 2,$
    $8x + 3y = 5.$

13. Plot the graphs of the pair of equations $x + y = 4$ and $2x + 2y = 8$.

Solution. You will find that the graph of each equation is represented by the same straight line. Can you find the reason? Such equations are called *equivalent equations.*

14. Plot the graphs of the pair of equations $x + 3y = 4$ and $2x + 6y = 10$.

Solution. In this case you will find that the graphs of the given equations are *parallel.* Hence the equations have no common solution. Such equations are called *inconsistent equations.* If we divide both sides of the second equation by 2, we obtain $x + 3y = 5$. How does this fact explain the name given to these equations?

NOTE. From Exercise 14 we infer that pairs of simultaneous linear equations have no common solution if they are *inconsistent.* The graphs of such equations are parallel lines. Equations which have distinct graphs, thus expressing different relations between the same unknowns, are called *independent equations.*

2. ALGEBRAIC SOLUTION OF PAIRS OF LINEAR EQUATIONS

132. Solution by Addition or Subtraction. The graphic method of solving pairs of equations is not always convenient. The solution may involve fractions or large numbers which it would be difficult to locate on a sheet of squared paper. A pair of independent linear equations in two unknown quantities can be solved algebraically by combining them in such a way that an equation in *one* unknown is obtained. This process is called *elimination.*

For example, consider the pair of equations

$$2x + y = 8,$$
$$3x - y = 7.$$

By *adding* the two left-hand members and the two right-hand members, we obtain the equation $5x = 15$. Hence $x = 3$. Substituting this value of x in either of the given equations, we find that $y = 2$. The solution (3, 2) may then be checked by substitution in the original equations. If the coefficients of y had both been *positive,* we should have eliminated y by *subtraction.*

When the coefficients of the unknown which is to be eliminated are *not* of the same absolute value, it becomes necessary to make these coefficients alike by multiplication before elimination is possible. This process is shown in the example which follows.

Illustrative Example. Solve the pair of linear equations

$$3x + 2y = 27,$$
$$5x - 3y = 7.$$

Solution. Multiplying both members of the first equation by 3, and both members of the second equation by 2, we have

$$9x + 6y = 81,$$

and
$$10x - 6y = 14.$$

Applying Axiom 1 by adding equals to equals, we have

$$19x = 95;$$

hence
$$x = 5.$$

If this value of x is substituted in the first of the given equations, we have

$$15 + 2y = 27;$$

whence
$$y = 6.$$

Hence the required solution is $x = 5$ and $y = 6$.

Check. Substituting 5 for x, and 6 for y, in each of the given equations, we have

$$3x + 2y = 15 + 12 = 27,$$

and
$$5x - 3y = 25 - 18 = 7.$$

The solution may be arranged in brief form, as follows:

$$\begin{array}{ll|l}
(1) & 3x + 2y = 27. & \\
(2) & 5x - 3y = 7. & \\
(3) & 9x + 6y = 81 & M_3. \\
(4) & \underline{10x - 6y = 14} & M_2. \\
(5) & 19x \qquad = 95 & \text{Ax. } 1. \\
(6) & x = 5. & D_{19}.
\end{array}
\qquad
\begin{array}{ll|l}
\multicolumn{2}{l}{\text{Substituting } x = 5 \text{ in } (1),} & \\
(7) & 15 + 2y = 27. & \\
(8) & 2y = 12. & S_{15}. \\
(9) & y = 6. & D_2.
\end{array}$$

Hence $x = 5$ and $y = 6$.

Check.
$$(1)\ 15 + 12 = 27.$$
$$(2)\ 25 - 18 = 7.$$

This method may be summarized as follows:

1. *Multiply both sides of each equation by such numbers as will make the coefficients of one of the unknown quantities the same in absolute value.*
2. *If these coefficients have like signs, subtract one equation from the other; if they have unlike signs, add.*
3. *Solve the resulting equation and thus find the value of one unknown.*
4. *By substituting this value in one of the given equations, find the value of the other unknown.*
5. *Check the results by substituting in each of the given equations.*

Exercises. *Elimination by Addition or Subtraction*

Solve the following pairs of equations by addition or subtraction. Check each set of results.

1. $x + y = 10,$
   $x - y = 2.$
2. $x - y = 3,$
   $x + y = 11.$
3. $x + 2y = 12,$
   $x - 2y = 4.$
4. $4x + 7y = 14,$
   $2x - y = 4.$
5. $x + 4y = 13,$
   $x + 3y = 10.$
6. $3x + 2y = 24,$
   $2x + 3y = 26.$
7. $5a + 6b = 14,$
   $3a + 2b = 10.$
8. $10x + 7y = -5,$
   $5x - 2y = 25.$
9. $12x + 5y = 26,$
   $4x - 3y = 18.$
10. $15a + 8b = 31,$
    $5a - 4b = -3.$
11. $8c - 5d = -1,$
    $7c + 15d = -59.$
12. $8x + 9y = 7,$
    $32x - 3y = 15.$
13. $5a + 9b = 59,$
    $7a + 3b = 73.$
14. $7x + 2y = -11,$
    $-9x + 5y = -1.$

15. $13m + 9n = 31,$
$-m + 5n = 9.$

16. $3x = -2y,$
$5x + 6y = -16.$

17. $a = 5b,$
$3a + 4b = 19.$

18. $7m + 7n = 0,$
$9m - 9n = 18.$

19. $11r - s = 17,$
$7r - 4s = -6.$

20. $7u + 5v = 22,$
$-9u + 6v = 9.$

21. $5A + 6B = 280,$
$7A - 4B = 20.$

22. $.1x + 2y = 11,$
$.5x - 3y = -10.$

23. $.19x - .17y = .02,$
$.21x + .15y = .36.$

24. $mx + 6y = 16,$
$mx - 3y = 7.$

25. $ax + 5y = c,$
$ax - 5y = d.$

26. $ax + by = c,$
$-ax + by = d.$

133. Solution by Substitution. A second method of solving pairs of equations is known as the *substitution method.* In this case we find the value of one unknown *in terms of the other* from one of the given equations, and substitute this value for that unknown in the other equation. This process also eliminates one unknown. The solution can then be completed in the usual way.

Illustrative Example. Solve the pair of equations

$$(1)\ x + 3y = 18,$$
$$(2)\ 7x - 4y = 1.$$

Solution. Solve equation (1) for x in terms of y. That gives

$$x = -3y + 18.$$

Substituting this value of x in equation (2), we have

$$7(-3y + 18) - 4y = 1.$$

Then $-21y + 126 - 4y = 1.$ (Why?)

That is, $-25y = -125;$

whence $y = 5.$ (Why?)

Substituting 5 for y in equation (1), we have

$$x + 15 = 18,$$

or

$$x = 3.$$

Hence the required solution is $x = 3$ and $y = 5$.

Check. Use the method explained on page 270.

Try to summarize this process in your own words. This method is convenient only when it does not involve complicated fractions.

Exercises. *Elimination by Substitution*

Solve the following pairs of equations by the method of substitution. Check each set of results.

1. $x = y + 1,$
$3x + 5y = 27.$

2. $x = 3y + 1,$
$5x + 2y = 39.$

3. $x = 7y - 2,$
$3x - 2y = 13.$

4. $y = 4x + 1,$
$6x + 3y = 21.$

5. $y = 5x - 2,$
$3x - 2y = -10.$

6. $x = 3y + 2,$
$x = 6y - 4.$

7. $x = 5y + 1,$
$x = 7y - 5.$

8. $a - b = 4,$
$3a + 4b = 47.$

9. $x - 2y = 6,$
$7x - 5y = 60.$

10. $5a + b = 19,$
$7a + 2b = 26.$

11. $7m + n = 25,$
$5m - 3n = 55.$

12. $3r + 2s = 1,$
$4r - 3s = -10.$

13. $3x - 5y = 6,$
$3x = 6y + 3.$

14. $4x = 7y - 1,$
$8x = 15y - 5.$

15. $.20y = .09x + .24,$
$.40y = .31x - .04.$

16. $21x = 17y - 22,$
$3x = 34 - 5y.$

17. $9y = 7x - 14,$
$11y = 3x + 10.$

18. $3u + 4v = 0,$
$u - 7v = 25.$

19. $5a - 6b = 43,$
$3a + 2b = 9.$

20. $5c + 3d = 2,$
$15c - 9d = 0.$

21. $x + 2y = 7x - y,$
$5x - 2y = 1.$

22. $3x - 2y = 4y - 3,$
$5y = 3x + 1.$

23. $8 + x + y = 4x,$
$20 - 6y = 5x - 1.$

24. $9 + 2x = 32 - 3y,$
$3x = 4y - 8.$

25. $3x + 2y = 4x,$
$5 - 2y = x - 7.$

26. $6x = 5y,$
$7x - 6y = -2.$

3. PROBLEMS

134. Number-Relation Problems. By using two letters it is often possible to solve a problem involving two or more unknown quantities more easily. The following example will serve to illustrate this fact:

Illustrative Example. The sum of two numbers is 56, and their difference is 14. Find the numbers.

Solution 1. Using One Letter. Let n represent the larger number. Then $n - 14$ represents the smaller number.

Hence $$n + n - 14 = 56.$$

Solving this equation, we find that
$$n = 35;$$
whence $$n - 14 = 21.$$

Check. $$35 + 21 = 56,$$
and $$35 - 21 = 14.$$

Solution 2. Using Two Letters. If x represents the larger number and y the smaller number, we have
$$x + y = 56,$$
and $$x - y = 14.$$

By adding these equations, we find that
$$2x = 70;$$
whence $$x = 35.$$

Substituting 35 for x in the first equation, we have

$$35 + y = 56;$$

whence $$y = 21.$$

Check. $$35 + 21 = 56,$$

and $$35 - 21 = 14.$$

NOTE. Observe that the second method enables us to *translate* the statements of the problem into algebraic form more easily than the first.

Exercises. *Number-Relation Problems*

1. The sum of two numbers is 9, and their difference is 3. Find the numbers.

2. The difference between two numbers is 2, and their sum is 4. Find the numbers.

3. The sum of two numbers is 1735, and they differ by 555. What are the numbers?

4. The sum of two numbers is 1000. If twice the first number is added to three times the second, the sum is 2222. Find the numbers.

5. Find two numbers such that their sum is 54 and the first is five times the second.

6. Separate 100 into two parts such that the larger is seven times the smaller.

7. Separate 500 into two parts such that the larger exceeds three times the smaller by 200.

8. The sum of two numbers is 95, and five times the larger exceeds three times the smaller by 147. What are the numbers?

9. Find two numbers differing by 12, whose sum is equal to four times their difference.

10. The sum of two numbers is a, and their difference is b. What are the numbers?

11. The difference between two numbers is 24, and seven times the smaller number exceeds the greater by 30. Find the numbers.

135. Digit Problems. The fingers of the hand have been and still are used in primitive ways of counting. It is not surprising, therefore, that the term *digit*, from the Latin word for "finger," has been applied to the figures 1, 2, 3, 4, 5, 6, 7, 8, and 9. Many interesting problems concerning the digits of arithmetical numbers can easily be solved by using two letters. It is necessary only to remember that a number such as 457 does *not* mean $4 + 5 + 7$, but $400 + 50 + 7$, or $4 \times 100 + 5 \times 10 + 7$.

Illustrative Example. In a number of two digits the tens' digit is 7 more than the units' digit, and the sum of the digits is 11. Find the number.

Solution. Let t represent the tens' digit and u the units' digit. We then have the equations

$$t - u = 7,$$

and

$$t + u = 11.$$

Solving, we find that

$$t = 9,$$

and

$$u = 2.$$

Hence the required number is 92, a result which can be checked in the usual way.

Exercises. *Digit Problems*

1. Find a number of two digits such that the tens' digit is twice the units' digit, the sum of the digits being 9.

2. Find a number of two digits such that the tens' digit exceeds the units' digit by 1, the sum of the digits being 13.

3. In a number of two digits the units' digit is four times the tens' digit. The difference of the digits is 6. Find the number.

4. In a number of two digits the tens' digit is 5 more than the units' digit, and the number is 3 more than ten times the tens' digit. Find the number.

Suggestion. If we use the letters t and u as explained above, the number is represented by $10\,t + u$.

5. In a number of two digits the sum of the digits is 14. If the digits are interchanged, the resulting number is 18 less than the original number. Find the number.

6. In a number of two digits the units' digit is the larger, being equal to three times the difference between the digits. If the order of the digits is reversed, the number is increased by 18. Find the number.

7. A number of two digits is equal to seven times the sum of its digits. If the digits are reversed, the resulting number is 36 less than the original number. What is the number?

8. Two numbers, each having two digits, differ by 36. The same digits appear in each number. If the smaller number is increased by its tens' digit plus four times its units' digit, the result is 100. What are the numbers?

9. A number consisting of two digits is seven times as large as the sum of the digits. If the order of the digits is reversed and the resulting number added to the original number, the sum is 66. Find the number.

10. In a number of three digits the units' digit is three times the hundreds' digit, and the tens' digit is 3 less than the units' digit. If the sum of the digits is 18, find the number.

136. Miscellaneous Problems. Many other types of problems can be solved more readily by using more than one letter to represent the unknown quantities. In order that you may become more skillful in analyzing problems of various kinds, the first twenty of the following exercises are left unclassified.

Exercises. *Miscellaneous Problems*

1. The perimeter of a rectangle is 40. Twice the length equals three times the width. Find the dimensions.

[illegible]

2. At a picnic John paid 35 cents for two sandwiches and a glass of milk. Fred paid 25 cents for one sandwich and two glasses of milk. How much did one sandwich cost? one glass of milk? [illegible]

3. Two sons together inherit $3000. If one gave to the other $250 of his inheritance, they would then have equal amounts. How much does each inherit?

4. One year ago a father was three times as old as his son, while 9 years ago he was five times as old. How old is each now?

5. At a student election in which 386 votes were cast, John was elected president of his class by a majority of 134 votes. How many of his classmates voted for him? How many did not?

6. A sedan costs $175 more than a touring car. Three of these touring cars cost only $150 more than two of the sedan cars. Find the price of each.

7. Two angles of a triangle are equal. The third angle exceeds either of the other two by 30 degrees. How many degrees are there in each angle?

8. The sum of two angles is 92 degrees, and their difference is 28 degrees. How large is each angle?

9. In London the longest day is 10 hours 2 minutes longer than the shortest night. How long is the longest day and how long is the shortest night in London?

10. A huckster has two baskets of apples, the first of which contains 51 apples more than the second. If the first basket contained three times as many apples as it does, and the second seven times as many as it does, the first would contain only 5 apples more than the second. How many apples are there in each basket?

11. Two pipes together furnish 224 gallons of water if each runs for 8 minutes. If the first runs 4 minutes longer and the second 3 minutes less, they furnish 280 gallons of water together. How much water does each pipe yield per minute?

12. I can buy 4 pounds of coffee and 2 pounds of tea for $4.20, while 5 pounds of coffee and 1 pound of tea would cost $3.90. Find the cost per pound of coffee and of tea.

13. Five carpenters and two plumbers worked for a contractor on a certain day. Their total wages for that day were $74. The next day seven carpenters and three plumbers appeared for work, their total earnings being $106. Find the daily wages of a carpenter and those of a plumber.

14. How many pounds of 40-cent coffee and how many pounds of 60-cent coffee must be mixed to make 30 pounds worth 48 cents a pound?

15. The wave lengths of two broadcasting stations together are equal to 1061 meters. If the first is subtracted from twice the second, the result is 544 meters. Find each wave length.

16. Two parcels, weighing 7 pounds and 12 pounds, cost 20 cents and 30 cents postage respectively. They were sent from the same post office and to the same address. What was the charge for the first pound and the charge for each additional pound?

17. At a school entertainment the receipts from the sale of 1000 tickets were $450. If each adult paid 75 cents and each child paid 25 cents, how many adults bought tickets, and how many children?

18. A telegram of 20 words cost 67 cents, whereas one of 32 words, sent to the same address, cost 97 cents. What was the rate for the first 10 words and the rate for each additional word?

19. A man had to pay $90 for 5 tons of hard coal and 2 tons of soft coal. His neighbor, who ordered 7 tons of hard coal and 3 tons of soft coal from the same dealer, paid $128 for his coal. What was the price of each quality of coal per ton?

20. In printing there is a first charge for typesetting, and then there is the charge per 100 copies of the material printed. If 500 school programs cost $4.80, and 800 programs cost $6.15, what was the charge for typesetting and the charge per 100 copies?

INTEREST PROBLEMS

21. A 4 per cent investment yielded $40 more annually than a 6 per cent investment. The sum of the two investments was $6000. How many dollars were invested at each rate?

22. A man decides to invest $20,000, a part in 6 per cent bonds and the remainder in 4 per cent bonds. If he desires

to have an annual income of $1100, how many dollars must he invest at each rate?

23. Two sums of money, namely, $10,000 and $7000, are invested at different rates of interest, yielding together an annual income of $950. If the first investment were $13,000 and the second were $5000, the yearly income would be $1030. Find the rates of interest.

24. A man invests $500 in real estate and $700 in railroad bonds. The two investments together yield $53 annually. If the rates of interest were interchanged, the income would be $55. Find the rates of interest on the two investments.

10 x + 25 x [illegible] = [illegible]

20 - d [illegible]

COIN PROBLEMS

25. A purse contains 20 coins, consisting of dimes and quarters. If these coins together amount to $3.35, how many dimes and how many quarters are there?

26. Frank said to John: "I have 15 coins in my pocket, made up of nickels and pennies. If the number of nickels were 1 more and the number of pennies 1 less, I should have 35 cents. How many nickels have I, and how many pennies?" Can you answer Frank's question?

27. John had $5.45 in quarters and dimes. He bought a pencil for 5 cents by using a quarter and receiving two dimes in change. Then the number of dimes was twice the number of quarters remaining. How many of each were there at first?

28. A man had 19 coins in his purse, made up of half-dollars, quarters, and dimes. The total amount of money in the purse was $4.70. If the number of quarters was twice the number of half-dollars, how many coins of each kind did the purse contain?

MOTION PROBLEMS

29. A boat can go 30 miles downstream in 2 hours, but only 20 miles upstream in 4 hours. Find the rate of the boat in still water, and also the rate of the current.

SUGGESTION. Let $x =$ the rate of the boat, and $y =$ the rate of the current. Then $x + y =$ the rate of the boat while going downstream, and $x - y =$ the rate of the boat while going upstream.

30. A canoeist can go downstream 24 miles in 3 hours, while the return trip takes 6 hours. Find the rate of the current and that of the canoeist.

31. A motor boat makes 8 miles per hour going upstream, and 20 miles per hour going downstream. What is the rate of the current, and what is the rate of the boat in still water?

32. An airplane covered a distance of 245 miles in 1 hour and 45 minutes while going *with* the wind. The return trip *against* the wind, over the same course, took 4 hours

and 5 minutes. What was the speed of the plane and the velocity of the wind?

33. The fastest commercial passenger service from New York to San Francisco completes the entire journey of 3520 miles in 48 hours, using an airplane by day and express trains by night. One third of the total time is spent on the train. If the express train travels half as fast as the airplane, find the average speed of the train and of the airplane.

137. Summary. In this chapter you have learned

(1) How to plot points on squared paper.

(2) How to draw the graph of a linear equation.

(3) How to solve pairs of linear equations graphically.

(4) How to solve pairs of linear equations by addition or subtraction.

(5) How to solve pairs of linear equations by substitution.

(6) How to use two letters in solving number-relation problems, digit problems, and other applied problems.

Test on Chapter X

1. Locate the following points on squared paper:

A, $(3, 4)$; B, $(-3, 4)$; C, $(-3, -2)$; D, $(3, -2)$.

2. (1) Connect in succession the points plotted in Exercise 1.

(2) What kind of figure is $ABCD$?

(3) In this figure draw the diagonals AC and BD.

(4) What are the coördinates of P, the point of intersection of AC and BD?

Draw the graph of one of the following equations, showing also the table on which the graph is based:

3. $x + y = 6$.

4. $y = 3x$.

Solve one of the following pairs of equations graphically, and check the result:

5. $2x + y = 7$,
$x - y = 8$.

6. $x + y = 7$,
$3x - y = 1$.

Solve the following pairs of equations algebraically:

7. $5x + 3y = 35$,
$2x - y = 3$.

8. $4x + 5y = 7$,
$x = 1 - 2y$.

Write the equations which you would use to solve the following problems:

9. The sum of two numbers is 6, and their difference is -10. Find the numbers.

10. A number of two digits is such that the sum of the digits is 7. If the digits are interchanged, the resulting number is 27 larger than the original number. What is the number?

11. An ice-cream stand sold both 5-cent and 10-cent ice-cream cones. At the close of a day the cash register showed 185 sales, and receipts of $12.50. How many of each kind were sold?

Exercises. *Review*

1. Does 5 satisfy the equation $7x - 5(4 - x) = 10(x - 1)$?

2. Simplify $46 - 5(2x - 3) + 6(1 - x)$.

3. Multiply $x^2 - 5x + 7$ by $2x^2 + 3x - 1$. Check the work by letting $x = 2$.

4. Divide $6a^3 - 7a^2 + 14a - 8$ by $2a^2 - a + 4$.

5. Using the formula $s = \frac{n}{2}(a + l)$, find the value of s to the nearest tenth if $n = 11$, $a = 7.56$, and $l = 15$.

6. If x concrete blocks weigh y pounds, how much will 3 blocks weigh? n blocks?

7. If an automobile can go d miles in 6 hours, how far can it go in 5 hours? in x hours?

8. Henry was a years old n years ago. Represent his present age and his age x years hence.

9. If 20 is 8 more than $4a$, what is the value of $a^2 - 9$?

10. What value must n have in the equation $3x - 2n = 0$ if 4 is a root of the equation?

11. If x diminished by 4 equals $3x$ divided by 5, what is the value of x?

12. What is the value of x at the point where the graph of $y = 2x - 6$ crosses the x-axis?

13. What is the Rule of Signs for Multiplication?

14. Find the value of each of the following:

(1) $3^2 \times 4^3$. (2) $(-1)^2$. (3) $(-2)^5$. (4) $-3(-3)^3$.

15. If $x = 4$, what is the difference in value between x^4 and $4x$? between $-5x^2$ and $5(-x)^2$?

16. From a rod x yards in length a piece d feet long is cut off at one end, and a piece n inches long is cut off at the other end. What is the length, in inches, of the remaining piece?

17. In a classroom there are 35 desks, some single and some double. The seating capacity of the room is 49. How many desks of each kind are there?

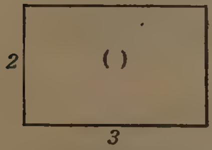
CHAPTER XI

SPECIAL PRODUCTS AND FACTORING

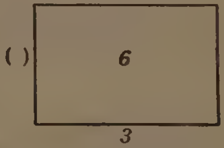
138. Special Products. In the multiplication of algebraic expressions certain combinations occur so frequently that it will save a great deal of time if the results can be found by a shorter method. We shall see in the following pages that some of these products may be obtained quickly by the use of special rules, and hence they are usually referred to as *special products.*

For example, cases such as the following will be considered: $(x+3)(x+5)$, $(2x+1)(3x-5)$, $(2a+b)(2a-b)$, $(a+b)^2$, and $(2-b)^2$.

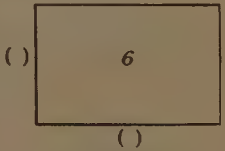
139. Factoring. The relation between multiplication, division, and factoring may be illustrated as follows:



Multiplication



Division



Factoring

In *multiplication* we are given the factors, and we are asked to find the product. In *division* we are given the product and one factor, and we are to find the other factor. In *factoring* we are given the product, and we are asked to find the factors. Hence *factoring is the reverse of multiplication.*

Factoring an expression is the process of finding two or more expressions whose product is the given expression.

In the present chapter some common cases of multiplication, and the reverse of each, are to be discussed. Only a few of the simpler and more frequently used methods of factoring will be presented.

140. Finding the Product of a Polynomial and a Monomial. We have previously learned how to multiply a polynomial by a monomial, by using the Rule of Distribution for Multiplication (see page 209). A review of some of the exercises given at that point will furnish an introduction to one of the simplest cases of factoring.

141. Factoring a Polynomial having a Common Monomial Factor. When a product has been obtained in the manner suggested in the preceding section, its factors will be a *monomial* and a *polynomial.*

The following arrangement will show the relationship involved:

MULTIPLICATION	FACTORING
$m(a+b) = ma + mb.$	$ma + mb = m(a+b).$

Since m is a factor of both ma and mb, it is called a *common factor.* Such a common factor can be discovered by a careful inspection of all the terms of the given polynomial. It will be observed that the other factor, $a+b$, is the result obtained by dividing the original polynomial, $ma+mb$, by the common factor, m.

Illustrative Example. Factor $5x^4 - 10x^3 + 15x^2$.

By inspection the common factor might be 5, x^2, or $5x^2$.

That is, $5x^4 - 10x^3 + 15x^2 = 5(x^4 - 2x^3 + 3x^2)$,

or $5x^4 - 10x^3 + 15x^2 = x^2(5x^2 - 10x + 15)$,

or $5x^4 - 10x^3 + 15x^2 = 5x^2(x^2 - 2x + 3)$.

The last form, in which the largest common factor, $5x^2$, has been chosen, is the one ordinarily used.

Hence, to factor a polynomial whose terms have a common monomial factor, we have the following rule:

1. *Find by inspection the largest common factor of all the terms of the polynomial.*
2. *Divide the polynomial by this common monomial factor.*
3. *Write the resulting quotient in a parenthesis preceded by the common monomial factor.*

Checking. All examples in factoring may be checked by multiplication; that is, in the case of any expression which has been factored the product of the factors must be equal to the given expression.

Exercises. *Common Monomial Factor*

Numbers 1 to 10, oral

In Exercises 1 to 10, the factors are given in Column A, and the corresponding products in Column B. Cover Column B with a piece of paper and read the products for the factors given; then reverse the process by covering Column A and factoring the products which appear in Column B.

COLUMN A	COLUMN B
1. $2(a+b)$	$=2a+2b$.
2. $3(w-3t)$	$=3w-9t$.
3. $a(m-r)$	$=am-ar$.
4. $4(2w-1)$	$=8w-4$.
5. $6x(x+2y)$	$=6x^2+12xy$.
6. $4a^2(3-5a)$	$=12a^2-20a^3$.
7. $n(r+s-2t)$	$=nr+ns-2nt$.
8. $2dw(2d^2-dw+2w^2)$	$=4d^3w-2d^2w^2+4dw^3$.
9. $\frac{1}{2}r(a+b+c)$	$=\frac{1}{2}ra+\frac{1}{2}rb+\frac{1}{2}rc$.
10. $.3x(x-2)$	$=.3x^2-.6x$.

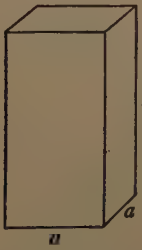
Factor each of the following, and check your results:

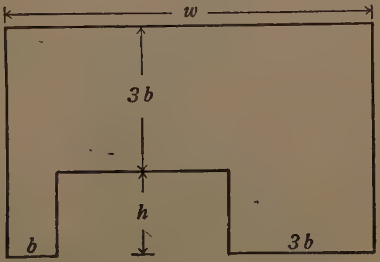
11. $5d - 5h$.	18. $7m - 14m^2 + 21m^3$.
12. $10x + 15y$.	19. $cx^2 + 4cx + 8c$.
13. $7cd - 14cf$.	20. $8rt^2 - 16rt + 12r$.
14. $P + PRT$.	21. $8e^3 + 4e^2 - 2e$.
15. $5a^2b - 25ab^2$.	22. $6\pi R_1s + 15\pi R_2s - 9\pi R_3s$.
16. $27n - 9$.	23. $d^2s^2 - 3ds^2 + 4d^3s - 12ds$.
17. $2\pi r^2 + 2\pi rh$.	24. $24m^2n - 6mn^2 + 36m^2n^2 - 42m^3$.

25. All the expressions appearing in the following table refer to rectangles. Find the missing expressions.

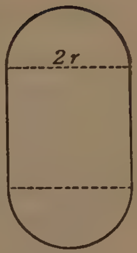
	LENGTH	WIDTH	PERIMETER	AREA
(1)	$3x + 5$	$6x$	$15x + 10$	$15x^2 + 30x$
(2)				$10x^2 - 2x$
(3)		$4n$		$4n^2 + 12n$
(4)	$12h + 3$		$24h + 10$	

26. Show that the total surface of the rectangular solid which is pictured below may be obtained by the formula $S = 2a(a + 2b)$.





27. Show that the area of the figure at the right above may be obtained by using the formula $A = b(3w + 4h)$. Find A if $w = 200$, $b = 30$, and $h = 50$.



28. The adjoining figure is made up of a square and two semicircles.

(1) Write a formula for its perimeter p.

(2) Rewrite this formula in factored form.

(3) Write a formula for the total area A.

(4) Rewrite in factored form the result obtained in (3).

Solve the following equations for x. Check each root.

29. $4x = 8a + 4b$.

30. $ax = a(b + c)$.

31. $cx = ac^2 - 5c^2$.

32. $(a + b)x = 5a + 5b$.

33. $2ax + bx = 2a^2 + ab$.

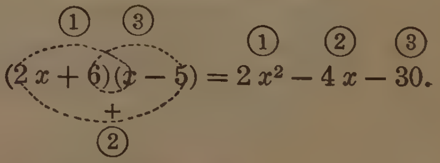
34. $r^2x - 5t^2x = 7ar^2 - 35at^2$.

142. Finding the Product of Any Two Binomials. Compare the usual method of multiplying $2x + 6$ by $x - 5$ with the short method shown below.

LONG METHOD

$$\begin{array}{rrr} 2x^2 & + \ 6 & \\ x & - \ 5 & \\ \hline 2x^2 & + \ 6x & \\ & -10x & - 30 \\ \hline 2x^2 & - \ 4x & - 30 \end{array}$$

SHORT METHOD (FOIL)



The dotted lines connect terms which are multiplied together. The small numbers in the circles show how the terms of the product may be obtained quickly.

From the above illustration we infer that, in general, the product of two binomials such as those given above is a trinomial, whose terms may be obtained as follows:

1. *The first term is the product of the first terms of the binomials.*
2. *The second term is found by combining the product of the two outer terms and that of the two inner terms.*

3. *The third term is the product of the* ***last terms*** *of the binomials.*

Key Words: **first terms, outer terms, inner terms, last terms.** (The rule stated above may also be remembered by the word **FOIL**, suggested by the first letters of the words *first, outer, inner, last.*)

Exercises. *Finding the Product of Two Binomials*

Numbers 1 to 10, oral

Supply the missing terms in the following products:

1. $(a+3)(a+4)=a^2+7a+(\quad\quad)$.
2. $(x+6)(x-3)=x^2+(\quad\quad)-18$.
3. $(3-5r)(2-r)=6-(\quad\quad)+5r^2$.
4. $(w+2h)(3w-5h)=(\quad\quad)+wh-(\quad\quad)$.
5. $(a-4b)(2a+6b)=(\quad\quad)-(\quad\quad)-24b^2$.

Find the following indicated products:

6. $(n+3)(n+2)$.
7. $(n-5)(n+2)$.
8. $(d-4)(d-5)$.
9. $(c-3)(c+12)$.
10. $(d+5)(d-3)$.
11. $(7+w)(8+w)$.
12. $(12-r)(5-r)$.
13. $(x+10)(x-6)$.
14. $(15-a)(4+a)$.
15. $(n+3)(2n+1)$.
16. $(3n-1)(n+4)$.
17. $(2n-3)(2n-1)$.
18. $(n-3)(4n+5)$.
19. $(a+b)(a+2b)$.
20. $(n-3s)(n+3s)$.
21. $(c^2-4)(c^2-12)$.
22. $(3+w^2)(7-w^2)$.
23. $(bh-12)(bh+7)$.
24. $(5+2cd)(4-3cd)$.
25. $(3d+1)(3d-3)$.
26. $(m+8)(m+8)$.
27. $(y-7)(y-7)$.
28. $(r+3d)(r-9d)$.
29. $(n^2-11p^2)(n^2+5p^2)$.

30. $(k - \frac{1}{2})(k - \frac{3}{4})$.
31. $(3n + 2)(5n - 1)$.
32. $(2 - 3n)(1 + 5n)$.
33. $(5 - 4a)(3 - 2a)$.
34. $(2a + \frac{3}{4})(4a + \frac{1}{4})$.
35. $(3w^2 - 2)(2w^2 + 1)$.
36. $(2x - 7y)(2x + 7y)$.
37. $(3n - m)(3n - 4m)$.
38. $(2k^2 + n)(2k^2 - 3n)$.
39. $(2xy + 3)(5xy - 6)$.
40. $(6 - 7n)(6 - 7n)$.
41. $(4a + 9b)(4a + 9b)$.
42. $(8 - 5ab)(3 - 2ab)$.
43. $(6x^2 - 5y^2)(5x^2 + 8y^2)$.
44. $(10c^2 - 3)(7c^2 - 4)$.
45. $(12 - 7r^2)(12 + 5r^2)$.
46. $(9s + 5t)(4s - 7t)$.
47. $(6m^2 + n^2)(10m^2 + 11n^2)$.
48. $(8c^2 - 3d)(4c^2 - 5d)$.
49. $(5b + 11c^2)(5b - 6c^2)$.
50. $(\frac{3}{4}d + 4c)(\frac{1}{2}d - 2c)$.
51. $(4h - .1)(h + .2)$.
52. $(7c - .8)(8c - .9)$.
53. $(6r - .7)(7r - .6)$.
54. $(.6c - 5)(.3c + 5)$.
55. $(1.2x^2 - .05)(.2x^2 + .7)$.

56. The following table gives the length and the width of several rectangles. Find the perimeter and the area in each case.

	LENGTH	WIDTH	PERIMETER	AREA
(1)	$x + 7$	$x + 3$		
(2)	$2n - 1$	$n + 8$		
(3)	$2d + 3$	$d - 1$		
(4)	$12s - 5$	$5s + 9$		

143. Factoring a Trinomial having the Form $ax^2 + bx + c$. If two binomials, such as $3x - 2$ and $x + 3$, are multiplied together, the product usually is a trinomial having the form $ax^2 + bx + c$. When such a binomial is to be factored, we can often obtain its binomial factors very easily by reversing the method explained in the preceding section.

Illustrative Example 1. Factor $3x^2 + 7x - 6$.

Solution. The binomial factors must evidently satisfy three requirements: (1) *the first terms must be factors of* $3x^2$; (2) *the last terms must be factors of* -6; (3) *these factors must be chosen in such a manner that when the product of the outer terms of the binomials is combined with the product of the inner terms, the result is* $+7x$.

Hence we must select the required binomials from possible combinations such as appear in the following list:

TRIAL FACTORS	CHECK FOR MIDDLE TERM
(1) $(3x+6)(x-1)$	$-3x+6x=+3x$
(2) $(3x-6)(x+1)$	$3x-6x=-3x$
(3) $(3x-1)(x+6)$	$18x-x=17x$
(4) $(3x+2)(x-3)$	$-9x+2x=-7x$
(5) $(3x-2)(x+3)$	$9x-2x=+7x$

Any one of the above combinations meets the first two requirements stated above, but only in the last arrangement do we obtain $+7x$, the required middle term. Hence

$$3x^2 + 7x - 6 = (3x-2)(x+3).$$

As you gain experience with this type of factoring, you will find that fewer trials will be necessary to discover the correct factors. Always remember that when the last term of the given trinomial is *positive*, its factors must have *like* signs; and when it is *negative*, its factors must have *unlike* signs.

Illustrative Example 2. Factor $10n^2 - 17n + 7$.

Solution. The first terms of the required binomials must be factors of $10n^2$, and the last terms must be factors of $+7$. Several possible combinations are shown below:

(1) $(5n+1)(2n+7)$
(2) $(5n-1)(2n-7)$
(3) $(10n-1)(n-7)$
(4) $(10n-7)(n-1)$

Since the last set furnishes the required middle term, $-17n$, it satisfies all three of the required conditions. Hence

$$10n^2 - 17n + 7 = (10n-7)(n-1).$$

Exercises. *Factoring Trinomials*

Numbers 1 to 10, oral

Supply the proper signs in each of the following sets of factors:

1. $b^2 + 8b + 12 = (b \quad 2)(b \quad 6)$.
2. $2n^2 + 5n - 12 = (n \quad 4)(2n \quad 3)$.
3. $d^2 - 8d + 16 = (d \quad 4)(d \quad 4)$.
4. $6y^2 + 29y - 5 = (6y \quad 1)(y \quad 5)$.
5. $10s^2 - 7s - 12 = (5s \quad 4)(2s \quad 3)$.
6. $6a^2 - 17ab + 5b^2 = (3a \quad b)(2a \quad 5b)$.

Factor each of the following trinomials:

7. $n^2 + 3n + 2$.
8. $w^2 - 6w + 5$.
9. $b^2 + 3b - 4$.
10. $y^2 - 2y - 3$.
11. $a^2 - 5a + 6$.
12. $c^2 + 2c - 8$.
13. $2x^2 + 7x + 3$.
14. $5R^2 - 11R + 2$.
15. $2d^2 + 5d + 2$.
16. $2d^2 - 3d - 2$.
17. $3c^2 - 10c + 3$.
18. $7m^2 - 5m - 2$.
19. $5 + 14K - 3K^2$.
20. $2 + 7x + 5x^2$.
21. $3 - 8y - 3y^2$.
22. $c^2 + 7c + 12$.
23. $a^2 - 8a + 12$.
24. $a^2 + 17a - 18$.
25. $H^2 + 5H - 24$.
26. $6n^2 + n - 2$.
27. $9x^2 - 18x + 5$.
28. $15r^2 - 4r - 11$.
29. $6n^2 + n - 5$.
30. $4x^2 - 16x + 15$.
31. $8W^2 + 18W + 9$.
32. $10m^2 + 7m - 6$.
33. $21a^2 - 31a + 8$.
34. $9 + 43D - 10D^2$.
35. $15s^2 + 22s + 8$.
36. $4 - 8y - 21y^2$.

37. $6x^2 + 11xy + 4y^2$.

38. $8a^2 - 18ab + 9b^2$.

39. $9x^2 + 12xy + 4y^2$.

40. $21n^2 - 37n + 12$.

41. $16m^2 + 32m - 9$.

42. $10x^2 - 27xy + 18y^2$.

43. $12 - 43d - 20d^2$.

44. $35b^4 + 61b^2 + 24$.

45. $40y^6 - 158y^3 + 63$.

46. $15a^4 + 4a^2b^2 - 35b^4$.

47. The first column of the following table gives the areas of certain rectangles. Find the items indicated.

	AREA	DIMENSIONS	PERIMETER
(1)	$d^2 + 8d + 15$		
(2)	$w^2 + 6w - 16$		
(3)	$2h^2 - 9h + 4$		
(4)	$6b^2 - 11b - 21$		

Timed Practice Exercises

Try to complete these exercises in 7 minutes.

In the following exercises the factors are given in Column A and the corresponding products in Column B. Cover Column B with a piece of paper and give the products suggested by Column A. Then reverse the process by covering Column A and factoring the products which appear in Column B.

COLUMN A COLUMN B

1. $(x + 7)(x + 3) = x^2 + 10x + 21$.
2. $(y - 5)(y - 2) = y^2 - 7y + 10$.
3. $(b + 4)(b - 3) = b^2 + b - 12$.
4. $(3n - 1)(n - 2) = 3n^2 - 7n + 2$.
5. $(2n + 5)(n - 1) = 2n^2 + 3n - 5$.
6. $(1 + 5a)(3 + a) = 3 + 16a + 5a^2$.

COLUMN A	COLUMN B
7. $(x+1)(4x+3)$	$=4x^2+7x+3.$
8. $(3x+2)(2x-3)$	$=6x^2-5x-6.$
9. $(4-c)(1-6c)$	$=4-25c+6c^2.$
10. $(5r-4s)(3r+2s)$	$=15r^2-2rs-8s^2.$

144. Factoring Completely. You will frequently find that one or more of the factors of an algebraic expression can be factored again.

For example,
$$3nx^2-15nx+18n=3n(x^2-5x+6)$$
$$=3n(x-2)(x-3).$$

This corresponds to the process of finding the *prime* factors of a number in arithmetic.

Thus, $12=3\times4=3\times2\times2.$

An expression which has no factors other than itself and unity is said to be *prime*.

The prime factors of an expression such as $6x^4+18x^3+12x^2$ would be written as follows:

$$6x^4+18x^3+12x^2=6x^2(x^2+3x+2)=3\cdot2\cdot x\cdot x(x+2)(x+1).$$

As a rule, however, only *polynomials* should be resolved into prime factors.

Thus, in the case of the above expression, it is customary to write

$$6x^4+18x^3+12x^2=6x^2(x+2)(x+1).$$

By "factoring completely," therefore, we shall understand that all polynomials which occur in the process of factoring shall be refactored, *if possible*, by any of the methods discussed in this chapter. Only answers in which the factoring has thus been completely carried out are accepted as correct.

Exercises. *Factoring Completely*

Supply the missing factors in each of the following:

1. $3n^2 - 12n + 9 = 3(\quad)(\quad)$.
2. $2x^2 - 4x - 30 = ?(x + 3)(x - 5)$.
3. $8a^3 + 44a^2 - 24a = 4a(\quad)(\quad)$.
4. $18r^3 - 30r^2 + 12r = ?(r - 1)(3r - 2)$.
5. $75y + 5y^2 - 10y^3 = ?(5 + 2y)(3 - y)$.
6. $24x^4 + 88x^3 + 80x^2 = 8x^2(\quad)(\quad)$.

Factor completely:

7. $5n^2 - 5n - 10$.
8. $3x^2 - 30x + 63$.
9. $na^2 + 7na + 6n$.
10. $2by^2 + 6by - 20b$.
11. $c - 4c^2 + 3c^3$.
12. $7sx^2 - 14sx - 21s$.
13. $6r^3 + 8r^2 + 2r$.
14. $18a^2b^2 + 21a^2b - 30a^2$.
15. $8n^3 + 20n^2 - 48n$.
16. $24h - 72h^2 + 30h^3$.
17. $8a^3b + 40a^2b^2 + 32ab^3$.
18. $12r^2 + 24r^3 - 96r^4$.
19. $48d^3 - 64d^4 + 16d^5$.
20. $16x^4y - 4x^3y^2 - 6x^2y^3$.

145. Squaring a Monomial. In certain problems it is necessary to multiply a monomial by itself.

For example, if the side of a square is $3w$, its area is

$$3w \cdot 3w = (3w)^2 = 9w^2.$$

Similarly,

$$(4a^3)^2 = 16a^6;\ (6a^2b^3)^2 = 36a^4b^6;\ \text{and } (-5xy^5)^2 = 25x^2y^{10}.$$

From these illustrations we infer the following rule:

The square of a monomial may be found by squaring the numerical coefficient and multiplying the result by each letter occurring in the monomial, giving to each letter an exponent twice as large as the one appearing in the original monomial.

Exercises. *Squaring Monomials*

Determine which of the following indicated squares have been found correctly and give the reason for your answer in each case:

1. $(x^3y^2)^2 = x^6y^4$.
2. $(-3\,n)^2 = 9\,n^2$.
3. $(5\,b^4)^2 = 25\,b^8$.
4. $(2\,dh^5)^2 = 4\,d^2h^{10}$.
5. $(-a^3b^4)^2 = a^6b^4$.
6. $(-4\,m^3n)^2 = 16\,m^5n^2$.
7. $(9\,wg^2y^3)^2 = 81\,w^2g^4y^6$.
8. $(-2\,rs^3)^2 = -4\,r^2s^6$.
9. $(7\,kyw)^2 = 49\,k^2y^2w^2$.
10. $(-12\,n^2)^2 = 144\,n^4$.
11. $(-9\,x^4yz^3)^2 = 9\,x^8yz^6$.
12. $(6\,c^2d^3)^2 = 36\,c^4d^9$.
13. $(\frac{1}{2}\,n^3)^2 = \frac{1}{4}\,n^6$.
14. $(15\,a^5b^4c^3)^2 = 225\,a^{10}b^8c^6$.
15. Why is the square of a monomial *always* positive?
16. Learn the following squares: $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, and so on, to $25^2 = 625$.

146. Finding the Square Root of a Monomial. Whenever two equal factors can be found for a monomial, it shows that the original monomial is a square. In that case either of the two equal factors is called the *square root* of the original monomial. Thus, since $9\,c^8$ may have $3\,c^4$ and $3\,c^4$ for its factors, it is a square, and $3\,c^4$ is the square root of $9\,c^8$.

But the equal factors $-3\,c^4$ and $-3\,c^4$ *also* produce $9\,c^8$, and hence $-3\,c^4$ is also the square root of $9\,c^8$. In this chapter, however, we shall have occasion to use only positive square roots.

Thus, the square root of 25 is 5, the square root of $16\,x^4$ is $4\,x^2$, and the square root of $4\,x^2y^6$ is $2\,xy^3$.

Hence we observe that *the square root of a monomial may be found by taking the square root of the numerical coefficient and multiplying this result by each letter occurring in the monomial, giving to each letter an exponent* ***half*** *as large as the one appearing in the original monomial.*

Exercises. *Square Root of Monomials*

1. Why may the square root of a number be either positive or negative?

2. How do the exponents in the square root of a monomial compare in size with those of the original monomial?

Find the positive square root of each of the following:

3. 36.
4. $25\, b^2$.
5. $9\, n^6$.
6. $16\, b^2c^2$.
7. $81\, r^4s^2$.
8. $169\, d^8$.
9. $\frac{9}{25}\, y^2$.
10. $\dfrac{64\, a^2}{225\, b^2}$.

Determine which of the following monomials are squares and, if possible, give the square root:

11. $100\, x^4$.
12. $25\, n^9$.
13. $-81\, b^2c^2$.
14. $225\, n^4d^8$.
15. $289\, x^{10}$.
16. $9\, ab^2$.
17. $\frac{25}{36}\, k^6$.
18. $\frac{2}{9}\, n^2$.

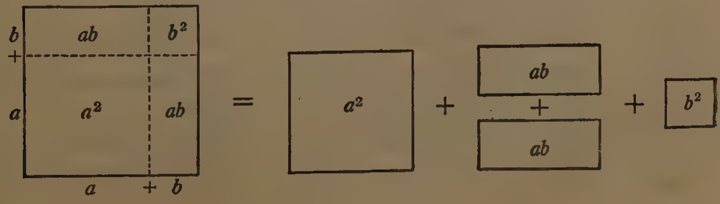
147. Squaring a Binomial. The square of a binomial such as $a + b$ may be indicated by writing $(a + b)^2$. Hence $(a + b)^2$ has the same meaning as $(a + b)(a + b)$. It follows, therefore, that

$$(a + b)^2 = (a + b)(a + b) = a^2 + 2\, ab + b^2. \qquad (1)$$

Similarly,

$$(a - b)^2 = (a - b)(a - b) = a^2 - 2\, ab + b^2. \qquad (2)$$

Since $(a + b)^2$ might represent the area of a square having for its side $a + b$, the result obtained in (1) above might be pictured as follows:



In a similar manner, the statement given in (2) may also be pictured.

As shown above, therefore, the square of a binomial may be found as follows:

1. *Square the first term.*
2. *Add twice the algebraic product of the two terms.*
3. *Add the square of the second term.*

Exercises. *Squaring Binomials*

Numbers 1 to 10, oral

Complete each of the following squares:

1. $(c - 2d)^2 = c^2 - 4cd + (\quad\quad)$.
2. $(2a + 5)^2 = 4a^2 + (\quad\quad) + 25$.
3. $(4m - 3n)^2 = 16m^2 - (\quad\quad) + 9n^2$.
4. $(n + 4d)^2 = (\quad\quad) + (\quad\quad) + 16d^2$.
5. $(y - \frac{1}{2})^2 = y^2 - (\quad\quad) + (\quad\quad)$.

Expand each of the following:

6. $(n + 3)^2$.
7. $(x - 5)^2$.
8. $(x + 9)^2$.
9. $(y + 6)^2$.
10. $(a - 1)^2$.
11. $(a - c)^2$.
12. $(r + s)^2$.
13. $(2r - 3)^2$.
14. $(3n + 5)^2$.
15. $(a - 3b)^2$.
16. $(4a + 3)^2$.
17. $(10x - y)^2$.
18. $(x - 8y)^2$.
19. $(3n^2 - 1)^2$.
20. $(cd + 6)^2$.
21. $(5r^2 - s^2)^2$.
22. $(3 + 10R)^2$.
23. $(9 - 2x)^2$.
24. $(6a - 11b)^2$.
25. $(a^3 + 12)^2$.
26. $(2s - 5t^2)^2$.
27. $(15b + 4ac)^2$.
28. $(K - \frac{1}{4})^2$.
29. $(n + \frac{2}{3})^2$.

Using the same method, find the value of the following:

30. $(23)^2$.

SUGGESTION. $(20 + 3)^2 = 400 + 120 + 9 = 529$.

31. $(21)^2$.
32. $(32)^2$.
33. $(61)^2$.
34. $(53)^2$.
35. $(19)^2$.
36. $(48)^2$.
37. $(17)^2$.
38. $(75)^2$.
39. $(27)^2$.

40. $(8\frac{1}{2})^2$.

SUGGESTION. $(8 + \frac{1}{2})^2 = 64 + 8 + \frac{1}{4} = 72\frac{1}{4}$.

41. $(3\frac{1}{2})^2$.
42. $(6\frac{1}{4})^2$.
43. $(9\frac{2}{3})^2$.
44. $(12\frac{1}{8})^2$.
45. $(10\frac{3}{4})^2$.
46. $(5\frac{2}{5})^2$.

47. A square whose side is n inches is enlarged so that the side becomes $(n + 2)$ inches. How much is its area increased?

48. A square piece of cardboard each of whose sides is s inches is to have 3 inches trimmed from each side. How much will its area be reduced?

49. The radius of a circle is r. How much will its area be increased if its radius becomes $r + 1$? $r + 5$? $r + n$?

148. Factoring a Trinomial Square. An expression having the form $a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$ is sometimes called a *trinomial square.* Such a trinomial may be factored either by the general method explained previously (see page 292) or by reversing the process discussed in the preceding section. Obviously, a trinomial square has these characteristics: (1) *it contains two terms which are squares, and these are positive;* (2) *the absolute value of the remaining term is always twice the product of the square roots of the other two terms.*

Hence, to obtain one of the two equal factors of a trinomial square, we have the following rule:

> *Write the square root of each of the two squares, and connect the results by the sign of the remaining term.*

Thus, $a^2 + 6a + 9 = (a + 3)^2$.

Also, $4x^2 - 12x + 9 = (2x - 3)^2$.

NOTE. In factoring a trinomial such as $x^2 - 10x + 25$, the result should ordinarily be given in the form $(x - 5)^2$, instead of $(x - 5)(x - 5)$.

Exercises. *Factoring Trinomial Squares*

Numbers 1 to 8, oral

Supply the missing terms in each of the following:

1. $a^2 + 6a + 9 = (a + ?)^2$.
2. $n^2 - 4n + 4 = (? - 2)^2$.
3. $16x^2 - 8x + 1 = (? - 1)^2$.
4. $9n^2 + 30ns + 25s^2 = (3n + ?)^2$.

Factor each of the following, if possible:

5. $x^2 - 12x + 36$.
6. $n^2 - 6n - 9$.
7. $49b^2 + 14b + 1$.
8. $81 + 36y + 4y^2$.
9. $16x^2 - 20xy + 25y^2$.
10. $25a^4 + 120a^2b^2 + 144b^4$.
11. $9m^2 - 3m + \frac{1}{4}$.
12. $4x^2 + 2.4x + .36$.
13. $100R^2 - 80R + 16$.
14. $256a^4 - 32a^2 + 1$.
15. $x^4 - 50x^2y^2 + 625y^4$.
16. $A^2 - 24AB + 144B^2$.

Factor completely each of the following:

17. $8x^2 + 8x + 2$.
18. $na^2 - 10na + 25n$.
19. $3b^3 + 18b^2 + 27b$.
20. $8c - 24c^2 + 18c^3$.
21. $16r^2 + 80rs + 100s^2$.
22. $150a^4 - 240a^3b + 96a^2b^2$.

Solve each of the following equations for x:

23. $(a+b)x=a^2+2ab+b^2$.
24. $ax+3x=a^2+6a+9$.
25. $cx-4x=c^2-8c+16$.
26. $rx+10r=r^2+5x+25$.
27. $bx-n^2=b^2+nx-2bn$.
28. $4r^2+4x=2rx+16r-16$.
29. $2ab-bx-b^2=a^2-ax$.
30. $c^2+2cd=cx-d^2+dx$.

149. Finding the Product of the Sum and the Difference of Two Terms. When we multiply together two binomials such as $a+b$ and $a-b$, the product of the outer terms equals the product of the inner terms in absolute value. But since these products have opposite signs, the final result is a *binomial* and not a trinomial. That is,

$$(a+b)(a-b)=a^2-b^2.$$

From this result we may readily infer the following rule:

> *The product of the sum and the difference of two terms equals the square of the first term used in each of the binomials minus the square of the second term.*

For example, $(a+4)(a-4)=a^2-16$.

Also, $(3x+5y)(3x-5y)=9x^2-25y^2$.

Exercises. *Finding the Product of the Sum and the Difference of Two Terms*

Numbers 1 to 10, oral

Find the product in each of the following cases:

1. $(x+y)(x-y)$.
2. $(n-3)(n+3)$.
3. $(5w+2)(5w-2)$.
4. $(3r-1)(3r+1)$.
5. $(10-d)(10+d)$.
6. $(1+7c)(1-7c)$.
7. $(6h+n)(6h-n)$.
8. $(4s+3n)(4s-3n)$.

9. $(a^2+11)(a^2-11)$.
10. $(ab-5c)(ab+5c)$.
11. $(9+2n^2)(9-2n^2)$.
12. $(3n+\frac{2}{3})(3n-\frac{2}{3})$.
13. $(12x-\frac{1}{2})(12x+\frac{1}{2})$.
14. $(x-.5)(x+.5)$.
15. $(1.5+w)(1.5-w)$.
16. $(\frac{1}{2}a-10)(\frac{1}{2}a+10)$.
17. $(\frac{3}{4}d+2c)(\frac{3}{4}d-2c)$.
18. $(.7s+\frac{1}{4}t)(.7s-\frac{1}{4}t)$.
19. $(\frac{2}{5}p-\frac{5}{2}d)(\frac{2}{5}p+\frac{5}{2}d)$.
20. $(2.5x+.03)(2.5x-.03)$.

EXPLANATION. The method used in Exercises 1–20 may be used to shorten certain arithmetical products. For example, in multiplying 34 by 26, we may write $34 \times 26 = (30+4)(30-4) = 900 - 16 = 884$.

Find the following products by the method indicated above:

21. 31 × 29.
22. 63 × 57.
23. 104 × 96.
24. 72 × 68.
25. 23 × 17.
26. 54 × 46.
27. 84 × 96.
28. 23 × 37.
29. 75 × 45.

30. A man building a house decided to make its length equal to the width. He found it necessary, however, to change the dimensions by increasing the length by 2 feet and diminishing the width by 2 feet. Was the area of the new ground plan larger or smaller than that of the original one, and how much?

150. Factoring the Difference of Two Squares. We have seen that $(a+b)(a-b)=a^2-b^2$. Hence the factors of a^2-b^2 are $a+b$ and $a-b$. That is,

$$a^2-b^2=(a+b)(a-b).$$

From this result we infer the following rule:

> *The difference of two squares may be factored by finding the square root of each of the squares in the given order and using as factors the sum and the difference of these roots.*

Exercises. *Factoring the Difference of Two Squares*

Numbers 1 to 12, oral

Factor each of the following expressions:

1. $h^2 - 9$.
2. $4n^2 - 81$.
3. $25 - t^2$.
4. $16w^2 - 1$.
5. $49 - 36c^2$.
6. $64n^6 - 121d^2$.
7. $m^2n^2 - 144$.
8. $R^2 - \frac{1}{4}$.
9. $25h^2 - \frac{4}{9}t^4$.
10. $81x^2 - .49$.
11. $.64n^4 - .09$.
12. $49a^2b^2 - 64c^2d^2$.

Factor each of the following expressions, if possible:

13. $121a^2 - 8$.
14. $36r^2 + s^2$.
15. $100n^4 - 9c^2$.
16. $4n^2 + 36$.
17. $225w^4 - 64$.
18. $2D^2 - 12$.
19. $16k^3 - 49$.
20. $144c^9 - 4d^2$.
21. $-81h^2 + 25t^2$.
22. $9c^3 - 3c$.

Factor completely each of the following:

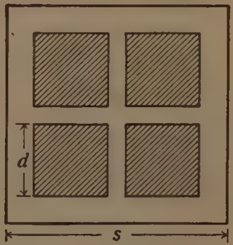
23. $2a^2 - 8$.
24. $n^3 - 9n$.
25. $27x^3 - 3x$.
26. $a^4 - b^4$.
27. $81n^4 - 1$.
28. $3x^4 - 48$.
29. $x^{2a} - y^{2a}$.
30. $(a + b)^2 - c^2$.
31. $x^2 - (y + z)^2$.
32. $(a + b)^2 - (a - b)^2$.
33. $a^2 + 2ab + b^2 - c^2$.
34. $c^2 + 2cd - r^2 + d^2$.
35. $x^2 + 2xy - 4c^2 + y^2$.
36. $4c^2 - 12cd - 16p^2 + 9d^2$.

37. From a square sheet of metal four equal squares have been cut. The dimensions are shown in the figure.

(1) Write a formula for finding the remaining area A.

(2) Rewrite in factored form the formula obtained in (1).

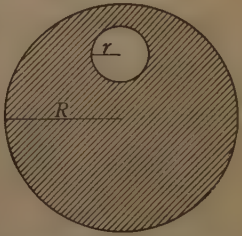
(3) Evaluate A if $s = 18$ inches and $d = 6$ inches, using (*a*) the formula as stated in (1), and (*b*) the formula as stated in (2). Which of the two processes is more convenient?



38. The radius of a circular tag is R. From this tag a circular piece, whose radius is r, has been cut, as shown in the adjoining figure.

(1) Write a formula for finding the remaining area A.

(2) Rewrite in factored form the formula obtained in (1).



39. The circular plate at the end of a boiler has a radius R. From this plate 64 circular openings, each having a radius r, have been cut for the fire tubes.

(1) Write a formula for finding the remaining area A.

(2) Rewrite in factored form the formula obtained in (1).

(3) Evaluate A if $R = 24$ inches and $r = 1.5$ inches.

40. A pupil "proved" that $2 = 1$ by writing these statements:

(1) $a^2 - a^2 = a^2 - a^2.$
(2) $(a + a)(a - a) = a(a - a).$
(3) $a + a = a.$
(4) $2a = a.$
(5) $2 = 1.$

Can you point out the error in this "proof"?

151. Summary of Special Products and Factoring. In the preceding sections of this chapter you have learned that

(1) Multiplication and factoring are reverse processes.
(2) The factors of a product may be checked by multiplication.
(3) Common monomial factors should always be removed first.

Comparison of Special Products and Factoring

SPECIAL PRODUCTS	FACTORING
1. *Polynomial by monomial.*	1. *Common monomial factor.*
$n(a + b - c) = na + nb - nc$.	$na + nb - nc = n(a + b - c)$.
2. *Product of two binomials.*	2. *Trinomials of form* $ax^2 + bx + c$.
a. General case.	*a. General case.*
$(a + 2b)(3a - b) = 3a^2 + 5ab - 2b^2$.	$3a^2 + 5ab - 2b^2 = (a + 2b)(3a - b)$.
b. Special case: the square of a binomial.	*b. Special case: trinomial square.*
$(a + b)^2 = a^2 + 2ab + b^2$. $(a - b)^2 = a^2 - 2ab + b^2$.	$a^2 + 2ab + b^2 = (a + b)^2$. $a^2 - 2ab + b^2 = (a - b)^2$.
3. *Sum and difference of two terms.*	3. *Difference of two squares.*
$(a + b)(a - b) = a^2 - b^2$.	$a^2 - b^2 = (a + b)(a - b)$.

Practice Exercises. *Review*

Factor completely:

1. $x^2 - 25$.
2. $6a - 24b$.
3. $9a^2 + 12a + 4$.
4. $2n^2 - 13n - 7$.
5. $6x^2 + 7xy + 2y^2$.
6. $9 - 16a^2$.
7. $x^2 - 9x$.
8. $5y^2 + 7y - 6$.
9. $n^2 - 12n + 36$.
10. $3x^2 - 3y^2$.
11. $1 - 6n + 9n^2$.
12. $2a^2 + 4ab + 2b^2$.
13. $x^3 + 9x$.
14. $b^2 - 2b - 35$.
15. $9x^2 - 12xy + 4y^2$.
16. $6a^2 + 19a + 10$.
17. $3c^2 - 9c + 6$.
18. $x^3 - 4x$.

19. $x^2 - 5x - 6$.
20. $a^2 - 25b^2$.
21. $2n^2 - 12n + 18$.
22. $8ax - 12bx + 20cx$.
23. $x^4 - y^4$.
24. $4c^2 + 20cd + 25d^2$.
25. $6a^3 - 9a^2b - 6ab^2$.
26. $3b^2 + b - 2$.
27. $6x^4 - 24x^3 - 30x^2$.
28. $b^2 - 14bc + 49c^2$.
29. $8x^2 - 72x^4$.
30. $m^8 - 16$.
31. $2 + 8x + 8x^2$.
32. $x^4 - 8x^2 - 9$.
33. $x^4 - 5x^2 + 4$.
34. $128 - 2R^2$.
35. $a^2b^2 - 4abc - 32c^2$.
36. $2x^6 - 7x^3 - 15$.
37. $8n^3 - 24n^2 + 18n$.
38. $9ab^3 - 9ab$.
39. $121K^2 - 1$.
40. $18a^2 + 5a - 2$.
41. $x^{2n} - 8x^n + 15$.
42. $4x^{2n} - y^{2n}$.
43. $n^2 - n + \frac{1}{4}$.
44. $3a^{2n} + 5a^nb^n - 2b^{2n}$.
45. $15y^8 + y^4 - 6$.
46. $6a^5 - 36a^3 + 48a$.

152. Using Special Products in Solving Equations. The solution of equations which contain indicated products may be shortened by using the rules for finding special products.

Illustrative Example. Solve the equation

$$(x + 3)^2 - (x + 3)(x - 2) = 25.$$

Solution.

(1)	$(x + 3)^2 - (x + 3)(x - 2) = 25.$	Given.
(2)	$(x^2 + 6x + 9) - (x^2 + x - 6) = 25.$	Why?
(3)	$x^2 + 6x + 9 - x^2 - x + 6 = 25.$	Rule of Distribution.
(4)	$5x + 15 = 25.$	S. T.
(5)	$5x = 10.$	S_{15}.
(6)	$x = 2.$	D_5.

Check. Substitute 2 for x in the *original* equation.

NOTE. The products obtained in step (2) should be inclosed in parentheses, as shown above. The parentheses may then be removed as in step (3), by applying the Rule of Distribution. These two steps will be helpful in avoiding mistakes in signs.

Exercises. *Using Special Products in Equations and Problems*

Solve each of the following equations and check each root:

1. $(x - 4)(x - 3) = x^2 - 2$.

2. $7x - (2x - 3)^2 = 10 - 4x^2$.

3. $4x^2 - (2x + 5)(2x - 5) = 5(x - 1)$.

4. $12x - 3x^2 = 4x - (x - 6)(3x - 2)$.

5. $(x + 6)^2 - 10 = (x + 8)(x - 2)$.

6. $(3x - 4)(3x + 4) = (9x - 2)(x - 2)$.

7. $(2x + 5)^2 - (4x - 1)(x + 3) = 1$.

8. $(x + 4)^2 - (x + 2)^2 = 0$.

9. $2(2x + 3) + (2x - 3)^2 = (2x + 3)(2x - 3)$.

10. $(3x - 2)^2 - 2(x + 8) = 3 + (x - 1)(9x + 5)$.

11. The difference between the squares of two consecutive whole numbers is 23. What are the numbers?

12. A man had a garden in the form of a square. He made each side 3 feet longer, and thus increased the area by 51 square feet. How long was the side of the original garden?

13. Can you find three consecutive even numbers such that the product of the last two is 32 more than the square of the first number?

14. The plans for a house called for a rectangular foundation which was 6 feet longer than it was wide. Later it was decided to make the house 3 feet shorter and 2 feet wider. This decreased the area of the ground plan by 18 square feet. What were the original dimensions of the foundation?

153. Solving Equations by Factoring. To solve an equation by factoring we shall have to make constant use of the following important principle:

> *If the product of two or more factors is zero, at least one of the factors must be equal to zero.*

Now consider the equation

$$x^2 - 8x + 15 = 0.$$

This equation may evidently be written in the form

$$(x - 5)(x - 3) = 0.$$

According to the principle stated above, either $x - 5 = 0$ or $x - 3 = 0$. But if

$$x - 5 = 0,$$

then $$x = 5;$$

and if $$x - 3 = 0,$$

then $$x = 3.$$

Substituting these values of x in the given equation, we have (1) $5^2 - 8 \times 5 + 15 = 0$, or $25 - 40 + 15 = 0$, or $0 = 0$;

and (2) $3^2 - 8 \times 3 + 15 = 0$, or $9 - 24 + 15 = 0$, or $0 = 0$.

That is, the given equation has *two* roots, 5 and 3.

An equation of the *second degree*, like $x^2 - 8x + 15 = 0$, is usually called a *quadratic equation*. You will find that *every quadratic equation has two roots*. Similarly, an equation of the *third degree*, such as $x^3 - 6x^2 + 11x - 6 = 0$, has *three roots*, and so on.

To solve an equation in one unknown by factoring, apply the following rule:

1. *Transform the given equation, if necessary, so that the right member is zero.*

2. *Factor the left member completely and set each factor which contains the unknown equal to zero.*
3. *Solve the resulting equations.*
4. *Check each solution.*

Illustrative Example. Solve the equation $x^2 + 4x = 21$.

Solution.	(1) $x^2 + 4x = 21$.	Given.
	(2) $x^2 + 4x - 21 = 0$.	Tr.
	(3) $(x + 7)(x - 3) = 0$.	Factoring.
	(4) If $x + 7 = 0$, $x = -7$.	S_7.
	(5) If $x - 3 = 0$, $x = 3$.	A_3.

Check. Substitute -7 and 3 in the given equation.

Exercises. *Solving Equations and Problems by Factoring*

Solve the following equations and check the roots:

1. $x^2 - 3x + 2 = 0$.
2. $x^2 + 2x - 15 = 0$.
3. $x^2 + 6x + 9 = 0$.
4. $2x^2 + 3x = 2$.
5. $x^2 + x = 12$.
6. $r^2 - 16 = 0$.
7. $y^2 + 12 = 8y$.
8. $y^2 - 5y = 0$.
9. $6x^2 - 24 = 0$.
10. $3x^2 - 2x = 40$.
11. $4x^2 + 30x = 54$.
12. $y^2 - 11y = -18$.
13. $9x^2 - 25 = 0$.
14. $x^2 - 5x - 24 = 0$.
15. $2x^2 + 3 = 5x$.
16. $\pi r^2 - 4\pi r + 4\pi = 0$.
17. $y^3 - 9y = 0$.
18. $4x^3 + 16x^2 = -12x$.
19. $(x - 2)(x + 1)(x - 5) = 0$.
20. $(2x - 1)(x + 3)(x + 4) = 0$.

EXPLANATION. In applied problems negative solutions sometimes have no meaning. In such cases they should be rejected.

21. The square of a certain number minus the number itself equals 56. Find the number.

22. Can you find a number such that the square of the number equals 12 more than the number itself?

23. Seven times a certain whole number equals 6 more than twice the square of the number. What is the number? (Can both roots be used in this case?)

24. Can you find two consecutive numbers whose product is 72?

25. Can you find two consecutive even numbers whose product is 48?

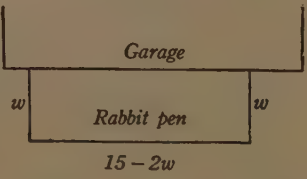
26. The sum of the squares of two consecutive numbers is 85. What are the numbers?

27. Find three consecutive odd numbers the sum of whose squares equals 83.

28. If from a square piece of cardboard a strip 4 inches wide is cut, the area of the remaining piece is 45 square inches. What is the length of a side of the original square?

29. A rectangle whose length is 5 inches more than its width contains 24 square inches. Find its dimensions.

30. A boy has 15 feet of wire fencing with which he wishes to build a rectangular rabbit pen. He is planning to build the pen along the side of the garage and so needs to use the wire fencing for three sides only, as shown in the adjoining figure. He wishes to have an inclosed area of 27 square feet. How long and how wide should he make the pen? Can the pen have more than one shape?



31. A 2-acre building lot is 4 rods longer than it is wide. What are its dimensions? (1 acre contains 160 square rods.)

32. From a square piece of cardboard a rectangular piece was cut which contained 30 square inches. To obtain this piece, one side of the cardboard was reduced by 3 inches, and the adjoining side by 2 inches. How long was a side of the original piece of cardboard?

154. Summary. In this chapter you have learned

(1) How to find the following special products:

- **(*a*) The product of any two binomials.**
- **(*b*) The square of any binomial.**
- **(*c*) The product of the sum and the difference of two terms.**

(2) How to deal with the following cases in factoring:

- **(*a*) Removing a common monomial factor.**
- **(*b*) Factoring a trinomial of the form $ax^2 + bx + c$.**
- **(*c*) Factoring a trinomial square.**
- **(*d*) Factoring the difference of two squares.**

(3) How to use special products and factoring in solving equations and problems.

Test on Chapter XI

1. Supply the missing terms in each of the following products:

(1) $(x + 3y)(x + 6y) = x^2 + 9xy + (\quad)$.
(2) $(3a - 2)^2 = 9a^2 - (\quad) + 4$.
(3) $(4m - 3n)(2m + 5n) = (\quad) + (\quad) - 15n^2$.

2. Insert the proper signs between the terms of the binomial factors in each of the following:

(1) $x^2 - 7x + 12 = (x \quad 4)(x \quad 3)$.
(2) $n^2 + 2n - 15 = (n \quad 3)(n \quad 5)$.
(3) $6y^2 - 5y - 4 = (2y \quad 1)(3y \quad 4)$.

3. Factor completely:

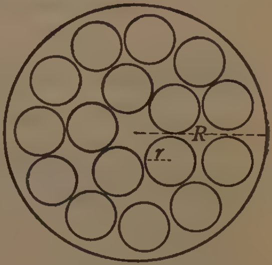
(1) $x^2 - 9$.
(2) $3a - 6b + 9c$.
(3) $x^2 - 3x - 18$.
(4) $8n^4 - 12n^3 + 4n^2$.
(5) $4R^2 - r^2$.
(6) $4 + 4h + h^2$.
(7) $2y^2 - 7y + 6$.
(8) $2\pi rh + 2\pi r^2$.
(9) $9x^2 - 30xy + 25y^2$.
(10) $9s - 36s^3$.
(11) $10a^2 + 26ab - 12b^2$.
(12) $20n^3 + 20n^2 + 5n$.
(13) $x^4 + 4x^2 - 5$.
(14) $D^2 - .81$.

4. Given a circular metal plate of radius R. From this plate 16 circular pieces, each of radius r, are to be cut.

(1) Write a formula for the remaining area A of the surface of the plate.

(2) Write in factored form the formula obtained in (1).

(3) Evaluate A if $R = 18$ and $r = 1.5$.



5. Solve the equation $(2x - 9)^2 = (x - 6)(4x + 3) - 6$.

6. Solve the equation $2x^2 + x = 6$.

7. The difference between the squares of two consecutive whole numbers is 29. What are the numbers?

8. A rectangle whose width is 2 inches less than its length contains 63 square inches. Find its dimensions.

CHAPTER XII

FRACTIONS

155. Algebraic Fractions. If we solve the formula $d = rt$ for r, we obtain $r = \frac{d}{t}$. The expression $\frac{d}{t}$ is an *algebraic fraction.*

In general, an expression such as $\frac{a}{b}$ is an *algebraic fraction.* It is read "a divided by b," or "a over b." As in arithmetic, a is called the *numerator* of the fraction, and b the *denominator.* The numerator and the denominator of a fraction are often called the *terms of the fraction.*

A fraction may be regarded as an *indicated quotient* in which the dividend is the numerator and the divisor is the denominator.

The fraction $\frac{a}{b}$ may also be indicated by writing $a \div b$ or a/b. When the numerator or the denominator is a polynomial, the horizontal dividing line serves as a sign both of division and of grouping. For example, $\frac{a-b}{c+d}$ has the same meaning as $(a-b) \div (c+d)$.

Exercises. *Using Algebraic Fractions*

1. Write the fraction formed by dividing 3 by 8; n by d; $n + 3$ by $2d + 5$.

2. Write the fraction $\frac{E}{R+r}$ in a different form.

3. How many pencils at 5 cents each can be bought for 60 cents? for n cents? for m dollars?

4. If one book costs a dollars, how many books can be bought for b dollars?

5. A schoolroom to which n pupils are assigned contains s cubic feet. How much air space does each pupil have?

6. If I paid c dollars for p bushels of potatoes, what was the cost per bushel?

7. One factor of n is a. What is the other factor?

8. What fractional part of one yard is x inches?

9. What fractional part of one year is a days?

10. If my daily expenses are e dollars, how long can I live on t dollars?

11. If the circumference of a wheel is c feet, how many times will it revolve in going one mile?

12. In how many hours can I cover a distance of $(p + q)$ miles at the rate of $(r + z)$ miles per hour?

13. A man can do a piece of work in four days. What part can he do in one day? in three days?

14. A carpenter took n days to complete a job. What part did he do in one day? in five days?

15. A farmer wishes to exchange x bushels of apples costing y cents a bushel for coffee costing n cents a pound. How many pounds of coffee should he receive?

16. I bought a yards of cloth at $3 per yard, and b yards at $2 per yard. What was the average price per yard?

17. Arrange the fractions $\frac{1}{2}$, $\frac{1}{5}$, $\frac{1}{4}$, $\frac{1}{20}$, $\frac{1}{3}$, $\frac{1}{100}$ in the order of size, from the largest to the smallest.

18. If x is a positive integer, which fraction is larger, $\frac{1}{x}$ or $\frac{1}{x+5}$?

19. If $n = 2$, arrange the following fractions in the order of size, beginning with the largest:

$$\frac{1}{2n},\ \frac{1}{3n+1},\ \frac{1}{2n-1},\ \frac{3}{5n+8},\ \frac{2}{7n-10}.$$

156. The Fundamental Principle of Fractions. It is often necessary to simplify algebraic fractions or to combine them by one of the four fundamental operations. As in the case of arithmetical fractions, we shall find it necessary to depend constantly on the following

Fundamental Principle

The numerator and the denominator of a fraction may be multiplied by the same expression, or divided by the same expression, without changing the value of the fraction.

For example, $\frac{3}{5} = \frac{3 \times 2}{5 \times 2} = \frac{6}{10}$, and $\frac{15}{24} = \frac{15 \div 3}{24 \div 3} = \frac{5}{8}$.

Similarly, $\frac{a}{b} = \frac{a \cdot m}{b \cdot m} = \frac{am}{bm}$, and $\frac{an}{bn} = \frac{an \div n}{bn \div n} = \frac{a}{b}$.

In all these cases we have merely changed the *form* of the given fractions, but not their value. *Two fractions which have the same value but not the same form are said to be equivalent.*

Exercises. *Applying the Fundamental Principle*

Numbers 1 to 9, oral

Explain how the second fraction was obtained from the first:

1. $\frac{3}{4} = \frac{15}{20}$.
2. $\frac{25}{75} = \frac{1}{3}$.
3. $\frac{3}{7} = \frac{15m}{35m}$.
4. $\frac{40}{64} = \frac{5}{8}$.
5. $\frac{4}{5} = \frac{12}{15}$.
6. $\frac{5}{6} = \frac{15bc}{18bc}$.

7. $\frac{a^2}{ab} = \frac{a}{b}$.

8. $\frac{4a^2b^3}{8a^3b^2} = \frac{b}{2a}$.

9. $\frac{5x}{15} = \frac{x}{3}$.

10. $\frac{2mx}{6} = \frac{mx}{3}$.

11. $\frac{x^4}{x^2y^3} = \frac{x^2}{y^3}$.

12. $\frac{xy^2}{x^2yz} = \frac{y}{xz}$.

13. $\frac{3x + 3y}{9} = \frac{x + y}{3}$.

14. $\frac{a + 3}{5} = \frac{4a + 12}{20}$.

15. $\frac{x + 1}{ab} = \frac{2b(x + 1)}{2ab^2}$.

In each of the following exercises supply the missing term:

16. $\frac{7}{8} = \frac{(\quad)}{24}$.

17. $\frac{5ab}{10ac} = \frac{(\quad)}{20a^2c}$.

18. $\frac{9bc}{27b^2c^2} = \frac{1}{(\quad)}$.

19. $\frac{30a^3b^3}{45a^2b^2} = \frac{2ab}{(\quad)}$.

20. $\frac{a}{a + b} = \frac{(\quad)}{2a + 2b}$.

21. $\frac{3x}{4x + 5} = \frac{9x}{(\quad)}$.

22. $\frac{a + b}{a - b} = \frac{(\quad)}{(a + b)(a - b)}$.

23. $\frac{a + 4}{a - 3} = \frac{a^2 - 16}{(\quad)}$.

24. $\frac{10}{20x + 30} = \frac{1}{(\quad)}$.

25. $\frac{12x^2}{8x^2 - 4x} = \frac{3x}{(\quad)}$.

157. Reducing Fractions to Lowest Terms. A fraction is in its lowest terms when the numerator and the denominator have no common factor except 1. As in arithmetic, it is often desirable to simplify an algebraic fraction by reducing it to its lowest terms. This may be done by applying the *Fundamental Principle of Fractions.* That is,

> *To reduce a fraction to its lowest terms, divide the numerator and the denominator by the product of all the common factors.*

In reducing a fraction to its lowest terms, the common factors may be crossed out, or *canceled.* Care should be taken to remember that when we divide an expression by itself, the result is 1.

Thus, $\frac{a}{a}=1,\quad \frac{2n}{2n}=1,\quad \frac{5x^3}{5x^3}=1,\quad \frac{a+b}{a+b}=1.$

Illustrative Example. Reduce to lowest terms $\frac{30ab^2c^3}{42a^3bc^3}$.

Solution.

$$\frac{30ab^2c^3}{42a^3bc^3}=\frac{\overset{1}{\cancel{2}}\cdot\overset{1}{\cancel{3}}\cdot 5\cdot\overset{1}{\cancel{a}}\cdot\overset{b}{\cancel{b^2}}\cdot\overset{1}{\cancel{c^3}}}{\underset{1}{\cancel{2}}\cdot\underset{1}{\cancel{3}}\cdot 7\cdot\underset{a^2}{\cancel{a^3}}\cdot\underset{1}{\cancel{b}}\cdot\underset{1}{\cancel{c^3}}}=\frac{5b}{7a^2}.$$

Exercises. *Reduction of Fractions*

Reduce each of the following fractions to its lowest terms:

1. $\frac{ab}{ac}$.
2. $\frac{ab}{a^2}$.
3. $\frac{5}{20x}$.
4. $\frac{30a}{6b}$.
5. $\frac{35x^2}{40x^3}$.
6. $\frac{a^2b}{ab^2}$.
7. $\frac{35r^3}{5r^4}$.
8. $\frac{4ac}{12a^2d^3}$.
9. $\frac{-24x^4y}{6y^2}$.
10. $\frac{b^2cd}{bc^2d^5}$.
11. $\frac{6ab^2y}{6b^2y^2}$.
12. $\frac{13a^2xy}{26a^2xy^2}$.
13. $\frac{-15a^3b^3}{-75a^3b^2}$.
14. $\frac{-72x^5}{84x^6}$.
15. $\frac{-26r^3s^2}{65rs^2}$.
16. $-\frac{90u^4v^3}{18u^3vw}$.
17. $\frac{55bc^2}{77cd}$.
18. $\frac{-a^2b^3c^4}{a^3b^4c^5}$.

Explanation. In simplifying fractions involving polynomials, it is necessary to guard against a very common error.

Suppose that the fraction $\frac{3}{4}$ were written in the form $\frac{2+1}{3+1}$. Would it be correct to cross out the 1's in $\frac{2+\cancel{1}}{3+\cancel{1}}$, thus obtaining $\frac{2}{3}$? Obviously this is not possible, for $\frac{3}{4}$ does *not* equal $\frac{2}{3}$. Similarly, we must *not* cross out the a's in $\frac{\cancel{a}+5}{\cancel{a}+6}$, thus obtaining $\frac{5}{6}$. In like manner, $\frac{4-1}{5-1}$ does *not* equal $\frac{4}{5}$.

In general, if the same number is added to, or subtracted from, the numerator and the denominator of a fraction, the value of the fraction is changed. Hence, only like *factors* can be removed, or canceled, from the numerator and the denominator of a fraction.

Thus, $\frac{a+b}{ac}$ is *not* the same as $\frac{\cancel{a}+b}{\cancel{a}c}$, or $\frac{b}{c}$. Likewise, $\frac{5a+3b}{4bc}$ does *not* equal $\frac{5a+3\cancel{b}}{4\cancel{b}c}$, or $\frac{5a+3}{4c}$, for b *is not a factor of the entire numerator, but only of one term of the numerator.*

19. Is $\frac{4}{5}$ the same as $\frac{4+3}{5+3}$? Explain.

20. Is $\frac{5-2}{7-2}$ the same as $\frac{5-\cancel{2}}{7-\cancel{2}}$, or $\frac{5}{7}$? Explain.

21. Observe that $\frac{16}{64}=\frac{1}{4}$. A boy wrote $\frac{1\cancel{6}}{\cancel{6}4}=\frac{1}{4}$. Was his method correct?

22. State which of the following equalities are correct:

(1) $\frac{5x}{5y}=\frac{x}{y}$; (2) $\frac{a-5}{b-5}=\frac{a}{b}$; (3) $\frac{5a}{5(c+d)}=\frac{a}{c+d}$.

23. Indicate in which of the following fractions it is correct to cross out the a's:

(1) $\frac{ax^2}{ay^2}$; (2) $\frac{a+y}{a+y}$; (3) $\frac{a-3}{a+7}$; (4) $\frac{ab}{a+7}$; (5) $\frac{9a}{am}$.

24. Indicate in which of the following fractions it is correct to cancel $(a+3)$:

(1) $\frac{4m(a+3)}{5p(a+3)}$; (2) $\frac{a+3}{a+3+x}$; (3) $\frac{a+3}{5(a+3)-b}$; (4) $\frac{(a+3)^2}{3(a+3)-d}$.

Simplify each of the following fractions:

25. $\frac{5(x+y)}{7(x+y)}$.

26. $\frac{9(a-b)}{18b(a-b)}$.

27. $\frac{21(a+b)}{7(a+3b)}$.

28. $\frac{6(a-b)^2}{9(a-b)}$.

29. $\frac{12(x-y)}{3b(x^2-y^2)}$.

30. $\frac{x^2+xy}{x^2-y^2}$.

31. $\dfrac{x^2 - y^2}{x^2 + 2\,xy - 3\,y^2}$.

32. $\dfrac{4\,a + 4\,b}{6\,a + 6\,b}$.

33. $\dfrac{am + bm}{ax + bx}$.

34. $\dfrac{2\,x - 4}{6\,x^2 - 24}$.

35. $\dfrac{2\,x - 6}{x^2 - 5\,x + 6}$.

36. $\dfrac{(x - y)(m + n)}{(x - y)^2(2\,m + n)}$.

37. $\dfrac{5(a - 2\,b)^2}{3\,a - 6\,b}$.

38. $\dfrac{an + bn - cn}{ap + bp - cp}$.

39. $\dfrac{a^2 - 7\,a + 12}{a^2 - 8\,a + 15}$.

40. $\dfrac{x^2 - 8\,x + 16}{x^2 - 9\,x + 20}$.

158. Multiplication of Fractions. In algebra the product of two or more fractions is found in the same way as in arithmetic.

For example, $\dfrac{2}{3} \times \dfrac{4}{5} = \dfrac{2 \times 4}{3 \times 5} = \dfrac{8}{15}$.

Similarly, $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{a \cdot c}{b \cdot d} = \dfrac{ac}{bd}$.

Also, $3 \times \dfrac{4}{5} = \dfrac{3}{1} \times \dfrac{4}{5} = \dfrac{12}{5}$.

In like manner, $m \cdot \dfrac{a}{b} = \dfrac{m}{1} \cdot \dfrac{a}{b} = \dfrac{am}{b}$.

That is, to find the product of two or more fractions, we have the following rule:

> *Write the product of the numerators over the product of the denominators, removing all factors that are common to the two products.*

Illustrative Example 1. Multiply $\dfrac{5\,a}{9\,b^2}$ by $\dfrac{36\,b^3}{15\,a^2}$.

Solution. $$\frac{5\,a}{9\,b^2} \cdot \frac{36\,b^3}{15\,a^2} = \frac{\overset{1}{\cancel{5}} \cdot \overset{4}{\cancel{36}} \cdot \overset{1}{\cancel{a}}\overset{b}{\cancel{b^3}}}{\underset{1}{\cancel{9}} \cdot \underset{3}{\cancel{15}} \cdot \underset{a}{\cancel{a^2}}\underset{1}{\cancel{b^2}}} = \frac{4\,b}{3\,a}.$$

The cancellation of common literal factors may usually be performed at sight.

Illustrative Example 2. Multiply $\frac{x^2-4}{x^2-5x+6}$ by $\frac{2x-4}{3x+6}$.

Solution. In examples of this type it is usually best to factor the numerator and the denominator of each fraction at once, and then to cancel the common factors. The work may be arranged as follows:

$$\frac{x^2-4}{x^2-5x+6}\cdot\frac{2x-4}{3x+6}=\frac{\overset{1}{\cancel{(x+2)}}\overset{1}{\cancel{(x-2)}}\cdot 2(x-2)}{(x-3)\underset{1}{\cancel{(x-2)}}\cdot 3\underset{1}{\cancel{(x+2)}}}=\frac{2(x-2)}{3(x-3)}.$$

Check. The work may be checked by numerical substitution. Care should be taken, however, that no value of x is chosen which causes any denominator to become zero, since division by zero is excluded (see page 223).

Exercises. *Multiplication of Fractions*

Numbers 1 to 10, oral

Find the product of the following fractions:

1. $\frac{3}{4}\times\frac{12}{15}$.
2. $\frac{5}{8}\times 16$.
3. $\frac{3}{7}\times 21$.
4. $20\times\frac{3}{4}$.
5. $28\times\frac{5}{7}$.
6. $36\times\frac{4}{9}$.
7. $4x\cdot\frac{1}{2}$.
8. $\frac{n}{7}\cdot 7$.
9. $\frac{5}{a}\cdot 4$.
10. $\frac{a}{b}\cdot\frac{b}{c}$.
11. $\frac{4c}{3d}\cdot\frac{3}{8}d$.
12. $16a^2b\cdot\frac{3}{8ab}$.
13. $10x^2\cdot\frac{5}{2x}$.
14. $\frac{6a}{5b}\cdot\frac{2}{3}b$.
15. $\frac{5a}{4}\cdot\frac{4a}{5}$.
16. $\frac{xy}{z}\cdot\frac{z}{x}$.
17. $\frac{ab}{5}\cdot\frac{10}{a}$.
18. $\frac{4a}{5}\cdot\frac{15}{8a}$.
19. $36x^2\cdot\frac{5y}{18x}$.
20. $\frac{24a^2}{9b^2}\cdot 27ab^2$.
21. $12\cdot\frac{x-3}{4}$.
22. $24\cdot\frac{2x-3}{6}$.
23. $60\cdot\frac{7-3x}{12}$.
24. $\frac{5x-14}{15}\cdot 45$.

25. $\frac{5r}{3st} \cdot \frac{7s^2t^3}{15r^3}$.

26. $\frac{2mv^2}{3u} \cdot \frac{21u^3}{2mv}$.

27. $\frac{3x}{7y^2} \cdot \frac{21y}{9x}$.

28. $\frac{4ab}{3b^2} \cdot \frac{9b^3}{8a^2b}$.

29. $\frac{4xyz}{7y} \cdot \frac{21}{2xy}$.

30. $\frac{5x}{9y^2} \cdot \frac{27y^5}{10x}$.

31. $\frac{x+y}{x-y} \cdot \frac{x-y}{15}$.

32. $\frac{x^2-y^2}{a+b} \cdot \frac{5a+5b}{7x+7y}$.

33. $\frac{x^2-121}{14} \cdot \frac{7}{x+11}$.

34. $\frac{a+b}{a-b} \cdot \frac{a^2-b^2}{a^2+ab}$.

35. $\frac{1-x^2}{y^3} \cdot \frac{x^2}{x+1}$.

36. $\frac{2x}{a-3b} \cdot \frac{2a-6b}{2xy}$.

37. $\frac{x-3y}{2y-x} \cdot \frac{8y-4x}{(x-3y)^2}$.

38. $\frac{xy}{a-b} \cdot (a^2-b^2)$.

39. $\frac{P+q}{P-q} \cdot \frac{4P-4q}{3P+3q}$.

40. $\frac{3a-9b}{15ax+45a} \cdot \frac{12a^3x+36a^3}{4a-12b}$.

41. $\frac{6x-30}{14am-6mx} \cdot \frac{7ax-3x^2}{3x^2-75}$.

42. $\frac{3x-7y}{v+4u} \cdot \frac{3x+7y}{v-4u}$.

43. $\frac{76}{x^2-16} \cdot \frac{3x-12}{57}$.

44. $\frac{4a^2-9b^2}{3x-2y} \cdot \frac{3x-2y}{2a-3b}$.

45. $\frac{x^3-9x}{xy^3+3y^3} \cdot \frac{x^2y^3z}{x^4z-3x^3z}$.

46. $\frac{a+b}{2l} \cdot \frac{3lm}{a^2+2ab+b^2}$.

47. $\frac{a^2-2ab+b^2}{(a+b)^2} \cdot \frac{a+b}{3a-3b}$.

48. $\frac{2c-8}{c^2-16} \cdot \frac{c^2+8c+16}{2c^2+5c-12}$.

49. $\frac{x^2-x-6}{x^2+5x+6} \cdot \frac{x^2-x-2}{x^2-2x-3}$.

50. $\frac{3a^2-2ax-x^2}{ax^2-3x^3} \cdot \frac{2a^2-3ax-9x^2}{2a^2+ax-3x^2}$.

51. If n apples cost a cents, how many cents will 12 apples cost? x apples?

52. If a pounds of sugar cost n cents, and butter costs 40 cents a pound, write the statement which shows that x pounds of butter cost as much as 10 pounds of sugar.

159. Division of Fractions. In algebra we divide by a fraction exactly as we do in arithmetic.

For example, $\frac{2}{3} \div \frac{4}{5} = \frac{2}{3} \times \frac{5}{4} = \frac{10}{12} = \frac{5}{6}$,

and $\frac{3}{4} \div 5 = \frac{3}{4} \times \frac{1}{5} = \frac{3}{20}$.

Also, $\frac{5}{9} \div 3\frac{1}{2} = \frac{5}{9} \div \frac{7}{2} = \frac{5}{9} \times \frac{2}{7} = \frac{10}{63}$.

In like manner, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$.

Also, $\frac{a}{b} \div n = \frac{a}{b} \cdot \frac{1}{n} = \frac{a}{bn}$.

That is, in dividing by an algebraic fraction, we may use the following rule:

Invert the divisor and proceed as in multiplication.

NOTE. Consider the fractions $\frac{a}{b}$ and $\frac{b}{a}$. Either of these fractions is called the *reciprocal* of the other. Hence the rule given above may be stated as follows: *To divide by a fraction, we may multiply by its reciprocal.*

Illustrative Example 1. Divide $\frac{4x^2y}{6ab}$ by $\frac{8xy}{3a^2b}$.

Solution.

$$\frac{4x^2y}{6ab} \div \frac{8xy}{3a^2b} = \frac{\overset{1}{\cancel{4}}\,\overset{x}{\cancel{x^2}}\,\overset{1}{\cancel{y}}}{\underset{2}{\cancel{6}}\,\underset{1}{\cancel{a}}\,\underset{1}{\cancel{b}}} \cdot \frac{\overset{1}{\cancel{3}}\,\overset{a}{\cancel{a^2}}\,\overset{1}{\cancel{b}}}{\underset{2}{\cancel{8}}\,\underset{1}{\cancel{x}}\,\underset{1}{\cancel{y}}} = \frac{ax}{4}.$$

Illustrative Example 2. Divide $\frac{2a+b}{2a+6}$ by $\frac{4a^2-b^2}{a+3}$.

Solution.

$$\frac{2a+b}{2a+6} \div \frac{4a^2-b^2}{a+3} = \frac{\overset{1}{\cancel{(2a+b)}}}{2\underset{1}{\cancel{(a+3)}}} \cdot \frac{\overset{1}{\cancel{(a+3)}}}{\underset{1}{\cancel{(2a+b)}}(2a-b)} = \frac{1}{2(2a-b)}.$$

Exercises. *Divison of Fractions*

Numbers 1 to 7, oral

Perform the following indicated divisions:

1. $\frac{4}{9} \div 2.$
2. $\frac{1}{2} \div 3.$
3. $\frac{5}{6} \div \frac{2}{3}.$
4. $\frac{4}{5} \div \frac{1}{2}.$
5. $1 \div \frac{a}{b}.$
6. $2n \div \frac{1}{2}.$
7. $6 \div \frac{2}{a}.$
8. $3x^2 \div 1\frac{1}{2}x.$
9. $\frac{a}{b} \div \frac{a}{c}.$
10. $\frac{5}{a} \div \frac{1}{a^2}.$
11. $\frac{ab}{cd} \div \frac{bc}{ad}.$
12. $1\frac{1}{2}x^3y \div 1\frac{7}{8}x^2.$
13. $\frac{15ab}{32xy} \div \frac{3a^2}{4x^2}.$
14. $\frac{a^3b^2}{7c^3d^4} \div \frac{a^4b^3}{14c^5d^6}.$
15. $\frac{11ax^3y^4}{36bp^2q} \div \frac{22ax^4y}{9bp^3q}.$

Perform the indicated operations in each of the following exercises:

16. $\frac{m-n}{xy^2} \div \frac{m-n}{x^3y}.$
17. $\frac{a+b}{x-y} \div \frac{b}{x-y}.$
18. $\frac{4a^2-b^2}{a+3} \div \frac{2a+b}{2a+6}.$
19. $\frac{(2a-b)^2}{x-2y} \div \frac{4ab^2-2b^3}{2x-4y}.$
20. $\frac{12r^2-27s^2}{12r-6s} \div (3s+2r).$
21. $\frac{2(x-3)}{x^2-4} \div \frac{x-3}{x+2}.$
22. $\frac{1}{x+y} \cdot \frac{x^2-y^2}{2x} \div \frac{ax-ay}{4x^3}.$
23. $\frac{3b^2}{bc-bc^2} \cdot \frac{1-c}{3b^2-3c^2} \div \frac{bc^2}{b+c}.$
24. $\frac{4a^2+4ab+b^2}{3lm} \div \frac{b+2a}{2l}.$
25. $\frac{2a}{9x^2-4y^2} \div \frac{2c}{9x^2-12xy+4y^2}.$

26. $\dfrac{a^2 - a - 12}{a^2 - 14\,a + 49} \cdot \dfrac{ax^2 - 7\,x^2}{a^2 - 9} \div \dfrac{a^2 - 6\,a + 8}{a^2 - 10\,a + 21}.$

27. $\dfrac{am + an}{m - n} \cdot \dfrac{m^2 - 2\,mn + n^2}{m^2 + 2\,mn + n^2} \div \dfrac{am^2x - an^2x}{mx + nx}.$

28. If a quarts of milk cost b dollars, how many quarts can be bought for 5 dollars? for n dollars?

29. If n pounds of coffee cost $(3\,c + 4)$ cents, and tea costs $(c + 10)$ cents a pound, how many pounds of coffee can be bought for the same amount of money as 10 pounds of tea?

160. Addition and Subtraction of Fractions having the Same Denominator. When algebraic fractions have the same denominator, they may be added or subtracted as in arithmetic.

Thus, $\dfrac{3}{12} + \dfrac{4}{12} = \dfrac{7}{12}$, and $\dfrac{7}{8} - \dfrac{5}{8} = \dfrac{2}{8} = \dfrac{1}{4}.$

Similarly, $\dfrac{a}{c} + \dfrac{b}{c} = \dfrac{a + b}{c}$, and $\dfrac{a}{c} - \dfrac{b}{c} = \dfrac{a - b}{c}.$

Exercises. *Addition and Subtraction of Fractions*

Add the following fractions, simplifying each result when possible:

1. $\dfrac{3}{8} + \dfrac{5}{8}.$

2. $\dfrac{1}{12} + \dfrac{7}{12}.$

3. $\dfrac{3}{20} + \dfrac{7}{20}.$

4. $\dfrac{5}{16} + \dfrac{9}{16}.$

5. $\dfrac{9}{32} + \dfrac{5}{32}.$

6. $\dfrac{1}{a} + \dfrac{3}{a}.$

7. $\dfrac{5}{n} + \dfrac{7}{n}.$

8. $\dfrac{a}{n} + \dfrac{b}{n}.$

9. $\dfrac{1}{8}\,x + \dfrac{3}{8}\,x.$

10. $\dfrac{a}{4\,b} + \dfrac{3\,a}{4\,b}.$

11. $\dfrac{7}{a + b} + \dfrac{5}{a + b}.$

12. $\dfrac{x}{x + y} + \dfrac{y}{x + y}.$

Subtract in each case, simplifying each result when possible:

13. $\frac{9}{10} - \frac{3}{10}$.

14. $\frac{5}{8} - \frac{1}{8}$.

15. $\frac{7}{32} - \frac{5}{32}$.

16. $\frac{7}{x} - \frac{5}{x}$.

17. $\frac{9}{n} - \frac{4}{n}$.

18. $\frac{a}{x} - \frac{b}{x}$.

19. $\frac{7a}{2b} - \frac{3a}{2b}$.

20. $\frac{x}{a+b} - \frac{y}{a+b}$.

21. $\frac{x}{x-y} - \frac{y}{x-y}$.

161. Changing Fractions to Equivalent Fractions; Lowest Common Denominator. When fractions do *not* have the same denominator, they cannot be combined by addition or subtraction until they have been changed to equivalent fractions having a common denominator. As in arithmetic, we shall avoid unnecessary work by using the *lowest common denominator.*

The expression *lowest common denominator* is usually represented by the abbreviation L.C.D.

To find the L. C. D. of two or more fractions, it is necessary to keep in mind the following important statement:

> *The L. C. D. of two or more fractions must contain each of the given denominators as an exact divisor. Hence it must contain each factor appearing in the denominators of the given fractions.*

1. When the given denominators have no common factor except unity, the L. C. D. is the *product of these denominators.*

For example, the L.C.D. of $\frac{1}{3}$ and $\frac{1}{4}$ is 12, and the L.C.D. of $\frac{1}{6}$ and $\frac{1}{5}$ is 30.

Similarly, the L.C.D. of $\frac{1}{a}$ and $\frac{1}{b}$ is ab, and the L.C.D. of $\frac{1}{a+b}$ and $\frac{1}{a-b}$ is $(a+b)(a-b)$, or $a^2 - b^2$.

2. If, however, the given denominators have common factors, the L. C. D. can be found by the following method:

(1) *Separate each denominator into its prime factors.*

(2) *Find the product of all the different prime factors, using each factor the greatest number of times it appears in any one denominator. This product is the L. C. D.*

Illustrative Example 1. Change $\frac{5}{6}$, $\frac{7}{15}$, and $\frac{9}{20}$ to equivalent fractions having the lowest common denominator.

Solution.

$$6 = 2 \cdot 3,$$
$$15 = 5 \cdot 3,$$
$$20 = 2 \cdot 2 \cdot 5.$$

Hence the L.C.D. $= 2 \cdot 2 \cdot 3 \cdot 5 = 60$.

That is, $$\frac{5}{6} = \frac{50}{60},\quad \frac{7}{15} = \frac{28}{60},\quad \text{and}\quad \frac{9}{20} = \frac{27}{60}.$$

Illustrative Example 2. Change $\frac{7x}{x^2 - 9}$, $\frac{3x}{2x - 6}$, and $\frac{x}{3x + 9}$ to equivalent fractions having the lowest common denominator.

Solution.

$$x^2 - 9 = (x + 3)(x - 3),$$
$$2x - 6 = 2(x - 3),$$

and $$3x + 9 = 3(x + 3).$$

Hence the L.C.D. $= 2 \cdot 3(x + 3)(x - 3) = 6(x^2 - 9)$.

It is evident, therefore, that the numerator and the denominator of the first fraction must be multiplied by 6 in order that its denominator may become $6(x^2 - 9)$.

That is, $$\frac{7x}{x^2 - 9} = \frac{7x}{x^2 - 9} \cdot \frac{6}{6} = \frac{42x}{6(x^2 - 9)}.$$

Similarly, $$\frac{3x}{2x - 6} = \frac{3x}{2(x - 3)} \cdot \frac{3(x + 3)}{3(x + 3)} = \frac{9x(x + 3)}{6(x^2 - 9)}.$$

Also, $$\frac{x}{3x + 9} = \frac{x}{3(x + 3)} \cdot \frac{2(x - 3)}{2(x - 3)} = \frac{2x(x - 3)}{6(x^2 - 9)}.$$

Hence, to change two or more fractions to respectively equivalent fractions, we have the following rule:

1. *Find the L.C.D. of the denominators of the fractions.*
2. *Multiply the numerator and the denominator of each fraction by those factors of the L.C.D. which are not factors of the denominator of that fraction.*

Exercises. *Finding the Lowest Common Denominator*

Numbers 1 to 12, oral

In each case change the given fractions to equivalent fractions having the lowest common denominator:

1. $\frac{1}{3}, \frac{1}{5}$.
2. $\frac{1}{8}, \frac{3}{4}$.
3. $\frac{1}{5}, \frac{1}{7}$.
4. $\frac{2}{3}, \frac{5}{8}$.
5. $\frac{3}{4}, \frac{5}{16}$.
6. $\frac{1}{2}, \frac{1}{3}, \frac{1}{5}$.
7. $\frac{1}{3}, \frac{1}{4}, \frac{1}{8}$.
8. $\frac{3}{5}, \frac{4}{9}, \frac{7}{15}$.
9. $\frac{1}{x}, \frac{1}{y}$.
10. $\frac{2}{a}, \frac{3}{b}$.
11. $\frac{5}{x}, \frac{15}{x^2}$.
12. $\frac{4}{ax}, \frac{7}{bx}$.
13. $\frac{x}{2}, \frac{x}{5}$.
14. $\frac{n}{7}, \frac{n}{9}$.
15. $\frac{a}{15}, \frac{b}{24}$. 120
16. $\frac{3a}{5}, \frac{7b}{8}, \frac{3c}{15}$. – 120
17. $\frac{5}{x^2}, \frac{7}{x}, \frac{8}{5}$. – 5x
18. $\frac{5a}{3a^2b}, \frac{3b}{4a^2c}, \frac{9}{5ac}$. 60a²c
19. $\frac{5}{x+1}, \frac{7}{x-1}$.
20. $\frac{9a}{a+b}, \frac{7b}{a-b}$.
21. $\frac{a-x}{7}, \frac{3a+x}{5}$.
22. $\frac{x+2}{x+3}, \frac{x-3}{x-2}$.
23. $\frac{7x}{x^2-9}, \frac{5x}{x+3}, \frac{x}{x-3}$.
24. $\frac{x+5}{x^2-16}, \frac{x-4}{x^2-x-20}, \frac{9}{x-4}$.

(x – 4)(x + 4)(x –

x² – 16

162. Addition and Subtraction of Fractions having Different Denominators. When two algebraic fractions which are to be combined by addition or subtraction do not have the same denominator, they must be changed to equivalent fractions having a common denominator. Their sum or their difference may then be found in the same way as in arithmetic.

Illustrative Example. Combine $\frac{a+2b}{5b}-\frac{2a-b}{3a}$.

Solution. The L.C.D. is $15ab$. (Why?)

Then
$$\frac{a+2b}{5b}-\frac{2a-b}{3a}=\frac{3a(a+2b)}{15ab}-\frac{5b(2a-b)}{15ab} \quad (1)$$
$$=\frac{3a(a+2b)-5b(2a-b)}{15ab} \quad (2)$$
$$=\frac{3a^2+6ab-10ab+5b^2}{15ab} \quad (3)$$
$$=\frac{3a^2-4ab+5b^2}{15ab}. \quad (4)$$

Check. Let $a=3$ and $b=2$. Substitute these values in the given expression and in the final result.

Then
$$\frac{a+2b}{5b}-\frac{2a-b}{3a}=\frac{3+4}{10}-\frac{6-2}{9}=\frac{7}{10}-\frac{4}{9}=\frac{23}{90}.$$

Also,
$$\frac{3a^2-4ab+5b^2}{15ab}=\frac{27-24+20}{90}=\frac{23}{90}.$$

NOTE. Observe that *parentheses* are necessary in (1) and (2) above, to indicate the required multiplications. In removing the parentheses, as shown in (3), *care should be taken that the law of signs is applied correctly*. Similar terms should then be combined and the result simplified, as shown in (4).

Therefore, to combine fractions by addition or subtraction, we may use the following rule:

1. *Change the fractions, if necessary, to equivalent fractions having their lowest common denominator.*
2. *Write the resulting numerators in succession over the L.C.D., using parentheses when necessary, each numerator being preceded by the sign of its fraction.*

3. *Remove all parentheses and write the algebraic sum of the resulting terms over the L.C.D.*
4. *Reduce the resulting fraction to its lowest terms.*

Exercises. *Addition and Subtraction of Fractions*

In each case combine the fractions as indicated:

1. $\frac{7}{15}+\frac{11}{30}-\frac{4}{5}.$
2. $\frac{2}{3}-\frac{1}{2}+\frac{1}{4}.$
3. $\frac{5}{9}+\frac{25}{36}-\frac{7}{24}.$
4. $\frac{3}{4}-\frac{5}{6}+\frac{13}{20}.$
5. $\frac{5a}{12}+\frac{5a}{6}-\frac{a}{3}.$
6. $\frac{7x}{9}+\frac{2x}{3}-\frac{5x}{18}.$
7. $\frac{7n}{30}+\frac{5n}{6}-\frac{4n}{15}.$
8. $\frac{5r}{14}-\frac{3r}{7}-\frac{r}{2}.$
9. $\frac{3}{a}+\frac{5}{2a}.$
10. $\frac{5}{7x}-\frac{3}{14x}.$
11. $\frac{3}{a}-\frac{2}{b}.$
12. $\frac{m}{5a}+\frac{p}{3b}.$
13. $\frac{7}{a^2}-\frac{5}{a^3}+\frac{2}{a}.$
14. $\frac{3}{x}+\frac{2}{y}-\frac{5}{2xy}.$
15. $\frac{x}{2y}-\frac{3}{y^2}+\frac{x}{3y}.$
16. $\frac{a}{m}+\frac{b}{n}+\frac{c}{p}.$
17. $\frac{a+1}{3}+\frac{a-1}{2}.$
18. $\frac{a+3}{5}-\frac{a+6}{10}.$
19. $\frac{5-3x}{8}-\frac{3-7x}{24}.$
20. $\frac{5b+3y}{18}-\frac{6b-8y}{36}.$
21. $\frac{3x}{4}+\frac{x+2}{x^2}.$
22. $\frac{x}{y}-\frac{x+c}{2y}.$ 2y
23. $\frac{7x-11y}{21}+\frac{7x-2y}{14}.$
24. $\frac{x-y}{2x}-\frac{x+y}{5x}.$
25. $\frac{a-b}{x}+\frac{x^2y}{a}.$
26. $\frac{2a+3}{6a^2}-\frac{4a+3b}{6a^2b}.$

27. Show that the sum of two "unit fractions," such as $\frac{1}{a}+\frac{1}{b}$, may be found by the formula $S=\frac{a+b}{ab}$. Using this formula, combine

(1) $\frac{1}{2}+\frac{1}{3}$; (2) $\frac{1}{5}+\frac{1}{7}$; (3) $\frac{1}{9}+\frac{1}{10}$; (4) $\frac{1}{2x}+\frac{1}{3y}$.

28. Make a formula for the difference D of two unit fractions, such as $\frac{1}{a}-\frac{1}{b}$. Apply this formula in combining

(1) $\frac{1}{2}-\frac{1}{3}$; (2) $\frac{1}{5}-\frac{1}{6}$; (3) $\frac{1}{8}-\frac{1}{15}$.

29. Show that the sum of two fractions, such as $\frac{a}{b}+\frac{c}{d}$, can be found by the formula $S=\frac{ad+bc}{bd}$.

Using this formula, combine

(1) $\frac{3}{5}+\frac{4}{7}$; (2) $\frac{5}{8}+\frac{2}{3}$; (3) $\frac{3}{4}+\frac{7}{10}$; (4) $\frac{x}{y}+\frac{r}{s}$.

30. Combine $\frac{3a^2}{a^2-1}+\frac{2a+1}{2a-2}-\frac{2a-1}{2a+2}$.

Solution. The L.C.D. is $2(a+1)(a-1)$.

Hence $\frac{3a^2}{a^2-1}+\frac{2a+1}{2a-2}-\frac{2a-1}{2a+2}$

$$=\frac{3a^2\cdot 2}{2(a+1)(a-1)}+\frac{(2a+1)(a+1)}{2(a+1)(a-1)}-\frac{(2a-1)(a-1)}{2(a+1)(a-1)} \quad (1)$$

$$=\frac{6a^2+(2a^2+3a+1)-(2a^2-3a+1)}{2(a+1)(a-1)} \quad (2)$$

$$=\frac{6a^2+2a^2+3a+1-2a^2+3a-1}{2(a+1)(a-1)} \quad (3)$$

$$=\frac{6a^2+6a}{2(a+1)(a-1)}=\frac{\overset{3}{\cancel{6}}a\overset{1}{\cancel{(a+1)}}}{\underset{1}{\cancel{2}}\underset{1}{\cancel{(a+1)}}(a-1)}=\frac{3a}{a-1}.$$

Check the result by letting $a=2$. In checking, avoid values of the letters which would reduce any denominator to zero (see page 223).

In each case combine the fractions as indicated and reduce the result to lowest terms:

31. $\frac{1}{a+5}+\frac{1}{a-5}$.

32. $\frac{1}{a-2}-\frac{1}{a+2}$.

33. $\frac{3}{a-4}-\frac{7}{a-3}$.

34. $\frac{9}{a+b}-\frac{6}{a}$.

35. $\frac{11}{3a-9}-\frac{18}{5a-15}$.

36. $\frac{4}{3x+9}+\frac{3x}{2x^2-18}$.

37. $\frac{3}{5(x-2)}-\frac{2}{7(x-2)}$.

38. $\frac{1}{a-b}-\frac{1}{a+b}$.

39. $\frac{x}{xy+y^2}-\frac{y}{x^2+xy}$.

40. $\frac{x}{2x+1}+\frac{y}{y+3}$.

41. $\frac{M}{M+m}+\frac{m}{M-m}$.

42. $\frac{a-b}{a+b}+\frac{a+b}{a-b}$.

43. $\frac{3x-1}{x+5}-\frac{x-2}{2x+3}$.

44. $\frac{x+y}{x-y}-\frac{2x-y}{2x+y}$.

45. $\frac{2x+3}{x-y}+\frac{5x-4}{3x-3y}$.

46. $\frac{2x-3}{2x-1}+\frac{5(x-1)}{4x^2-4x+1}$.

47. $\frac{3}{x+3}-\frac{x+4}{x^2+7x+12}$.

48. $\frac{x}{4+x}+\frac{x}{4-x}+\frac{4x^2}{16-x^2}$.

49. $\frac{3(5P-q)}{5P+q}-\frac{2(5P+q)}{5P-q}+\frac{50P^2+2q^2}{25P^2-q^2}$.

50. $\frac{1}{4x^2-12x+9}-\frac{1}{4x^2-9}$.

51. $\frac{2x}{3x-2}+\frac{x+3}{x+1}-\frac{2x^2-3x+1}{3x^2+x-2}$.

EXPLANATION. In arithmetic an expression such as $4\frac{2}{3}$ is often called a mixed number. Similarly, in algebra an expression such as $a+\frac{b}{c}$ is called a *mixed expression.*

Now, $4\frac{2}{3}$ means $4+\frac{2}{3}$. Hence $4\frac{2}{3}=\frac{4}{1}+\frac{2}{3}=\frac{12}{3}+\frac{2}{3}=\frac{14}{3}$.

Similarly, $a + \frac{b}{c}$ may be written $\frac{a}{1} + \frac{b}{c} = \frac{ac}{c} + \frac{b}{c} = \frac{ac + b}{c}$.

Also, $3x + 2 + \frac{4}{x}$ becomes $\frac{3x + 2}{1} + \frac{4}{x} = \frac{3x^2 + 2x + 4}{x}$.

Change into fractional form the following mixed expressions:

52. $3x + 1 + \frac{2}{x}$.
53. $4a - 5 + \frac{3}{a}$.
54. $6a - 2b - \frac{4}{3a}$.
55. $4 - 7x - \frac{3}{5x}$.
56. $x^2 + 2x + 1 + \frac{1}{x}$.
57. $3x^2 - 5x + 6 - \frac{3}{2x}$.
58. $4a^2 + 7ab - b^2 + \frac{5}{2a}$.
59. $a + b - \frac{ab}{a + b}$.
60. $x - 2y + \frac{7xy}{x + 3y}$.
61. $r - 3s - \frac{4s}{7r - 2s}$.

62. Given the fraction $\frac{a}{b}$, in which a and b are positive integers greater than 2. In the case of each of the following fractions indicate whether its value is greater than, less than, or equal to the value of the given fraction:

(1) $\frac{a}{b + 2}$; (2) $\frac{a}{b - 1}$; (3) $\frac{a + 10}{b}$; (4) $\frac{5a}{5b}$; (5) $\frac{a - 2}{b + 6}$.

63. If a and b are positive integers greater than 1, compare the value of the fraction $\frac{1}{a}$ with the value of

(1) $\frac{1}{ab}$; (2) $\frac{1}{a + b}$; (3) $\frac{1}{a^2}$; (4) $\frac{b}{a}$; (5) $\left(\frac{1}{a} + \frac{1}{b}\right)b$.

64. Show that the following statement is correct:

$$\frac{C}{D} = E + \frac{C - DE}{D}.$$

65. State whether the following formula is correct or incorrect:

$$\frac{C}{D} = \frac{C + DE}{D} - E.$$

163. The Three Signs of a Fraction. Since a fraction is an indicated quotient, the law of signs for division may be applied. In doing so, three signs must be considered: the sign of the numerator, the sign of the denominator, and the sign before the fraction, which is the sign of the quotient.

Now observe the following quotients, each of which is evidently equal to $+2$:

$$+\frac{+8}{+4}=+\frac{-8}{-4}=-\frac{+8}{-4}=-\frac{-8}{+4}=+2.$$

Similarly, $+\frac{+a}{+b}=+\frac{-a}{-b}=-\frac{+a}{-b}=-\frac{-a}{+b}.$

From these illustrations we infer the following principle:

> *The value of a fraction is not altered if we change* (1) *the sign of the numerator and the sign of the denominator, or* (2) *the sign before the fraction and also the sign of either the numerator or the denominator.*

In other words, we may change *any two of the three signs of a fraction* without changing its value.

It is important to remember that changing the sign of a polynomial numerator involves changing the sign of *each term* of the numerator. A similar precaution is necessary in the case of a polynomial denominator.

Changes of sign such as those mentioned under (1) above may also be explained by means of the *Fundamental Principle of Fractions.* For we can multiply or divide the numerator and the denominator of a fraction by -1.

For example, $+\frac{-8}{-4}=+\frac{(-8)(-1)}{(-4)(-1)}=+\frac{+8}{+4}.$

Similarly, $-\frac{+4}{-7}=-\frac{(+4)(-1)}{(-7)(-1)}=-\frac{-4}{+7}.$

In like manner, $-\frac{3}{4-x}=-\frac{3(-1)}{(4-x)(-1)}=-\frac{-3}{x-4}.$

Exercises. *Changing Signs in Fractions*

Numbers 1 to 12, oral

In each case explain why the indicated equality is correct:

1. $\frac{-3}{9}=-\frac{1}{3}$.
2. $\frac{7}{-14}=-\frac{1}{2}$.
3. $-\frac{3}{-7}=\frac{3}{7}$.
4. $\frac{6}{-24}=-\frac{1}{4}$.
5. $-\frac{-9}{-45}=-\frac{1}{5}$.
6. $-\frac{8pq^2}{-4pq}=2q$.
7. $-\frac{x}{3t}=\frac{-x}{3t}$.
8. $\frac{-y}{3-y}=\frac{y}{y-3}$.
9. $\frac{-x}{x-y}=\frac{x}{y-x}$.
10. $-\frac{2-x}{-1+x}=\frac{x-2}{x-1}$.
11. $\frac{x-3}{x}=\frac{3-x}{-x}$.
12. $-\frac{3a-b}{2a+b}=\frac{b-3a}{2a+b}$.

Reduce the following fractions to lowest terms:

13. $\frac{x-3}{6-2x}$.
14. $\frac{3x-12}{8-2x}$.
15. $\frac{3x-3y}{y^2-x^2}$.
16. $\frac{(d-c)^2}{c^2-d^2}$.
17. $\frac{14-7x}{x^2-4}$.
18. $\frac{x^2-7x+12}{16-x^2}$.

Perform the indicated operations in each case:

19. $\frac{4}{x^2-9}+\frac{2}{3-x}$.
20. $\frac{7}{a-b}+\frac{5}{b-a}$.
21. $\frac{2c}{c^2-9}-\frac{1}{3-c}$.
22. $\frac{1-x}{3-x}\cdot\frac{x^2-5x+6}{x^2-1}$.
23. $\frac{a-b}{2(c-b)}\cdot\frac{10(b-c)}{3(b-a)}$.
24. $\frac{a-4}{3-a}\div\frac{4-a}{a^2-4a+3}$.

164. Complex Fractions. The division of a fraction by a fraction may be indicated either horizontally or vertically.

That is,
$$\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}.$$

In general, a fractional expression which has one or more fractions either in its numerator or in its denominator, or in both, is called a *complex fraction.*

Thus, $\frac{\frac{2}{3}}{5}$, $\frac{m}{\frac{x}{y}}$, $\frac{a+\frac{b}{c}}{a-\frac{b}{c}}$ are complex fractions.

Complex fractions sometimes arise in the construction and the evaluation of formulas, and in the numerical checking of the algebraic processes.

For example, the engineering formula $h = \frac{bd^2}{\frac{w^2}{4}}$ contains a complex fraction. Again, in the case of the formula $r = \frac{d}{t}$, if $d = 40.5$ and $t = 3\frac{1}{4}$, we obtain $r = \frac{40.5}{3\frac{1}{4}}$.

A complex fraction may be simplified by carrying out the indicated processes in the usual way, as shown above. It is often more convenient, however, to *multiply the numerator and the denominator by the L.C.D. of all the fractions appearing in the given complex fraction.*

For example,
$$\frac{5\frac{1}{2}}{7\frac{1}{3}} = \frac{\frac{11}{2} \times 6}{\frac{22}{3} \times 6} = \frac{33}{44} = \frac{3}{4}.$$

Similarly,
$$\frac{\frac{a}{b}+3}{\frac{c}{d}-5} = \frac{bd\left(\frac{a}{b}+3\right)}{bd\left(\frac{c}{d}-5\right)} = \frac{ad + 3\,bd}{bc - 5\,bd}.$$

Exercises. *Complex Fractions*

Simplify each of the following complex fractions:

1. $\dfrac{2-\frac{1}{8}}{5}$.

2. $\dfrac{5}{8+\frac{1}{5}}$.

3. $\dfrac{6-\frac{3}{4}}{12}$.

4. $\dfrac{3+\frac{1}{5}}{4-\frac{1}{10}}$.

5. $\dfrac{x}{1+\frac{1}{2}}$.

6. $\dfrac{2x-\frac{3}{4}}{1-\frac{3}{4}}$.

7. $\dfrac{\frac{2}{3}}{2x-\frac{1}{3}}$.

8. $\dfrac{x}{1-\dfrac{1}{x}}$.

9. $\dfrac{\frac{4}{5}-\frac{7}{10}}{\frac{3}{20}+\frac{7}{25}}$.

10. $\dfrac{\frac{1}{2}+\frac{2}{3}+3}{\frac{3}{4}+\frac{1}{2}-1}$.

11. $\dfrac{\dfrac{m^2}{5}-5}{1+\dfrac{m}{5}}$.

12. $\dfrac{\dfrac{1+x}{3}}{1-\dfrac{1}{x^2}}$.

13. $\dfrac{m-\dfrac{4}{m}}{1+\dfrac{2}{m}}$.

14. $\dfrac{x-\dfrac{x^2}{y}}{1-\dfrac{x}{y}}$.

15. $\dfrac{\dfrac{m}{n}-\dfrac{n}{m}}{\dfrac{m}{n}+\dfrac{n}{m}}$.

16. $\dfrac{P}{P-\dfrac{q}{r}}$.

17. $\dfrac{\dfrac{a+b}{a-b}}{\dfrac{a-b}{a+b}}$.

18. $\dfrac{1+\dfrac{1}{x+1}}{1-\dfrac{1}{x+1}}$.

19. $\dfrac{\dfrac{a}{b}-2-\dfrac{3b}{a}}{\dfrac{a}{b}+\dfrac{3b}{a}-4}$.

20. Simplify the formula $K=\dfrac{\dfrac{a}{b}}{c-\dfrac{m}{b}}$.

21. Show that $x=4\frac{2}{3}$ satisfies the equation $\dfrac{x+2}{x-3}=4$.

22. Show that $x=10.1$ satisfies the equation

$$\frac{2x+.3}{x-6}=5.$$

Practice Exercises

Time, 40 minutes

Supply the missing term in each case:

1. $\frac{ab}{2x} = \frac{(\ \ )}{6x}$.

2. $\frac{3x^3}{7y^3} = \frac{15x^5}{(\ \ )}$.

3. $\frac{2a^2x + 3ax^2}{2axy - 3axz} = \frac{2a + 3x}{(\qquad)}$.

Reduce to lowest terms:

4. $\frac{3ab^3}{6ab^5}$.

5. $\frac{6a + 12}{5a + 10}$.

6. $\frac{bx - 4b}{bx^2 - 2bx - 8b}$.

Perform the indicated operations:

7. $\frac{3}{a} + \frac{2}{b}$.

8. $\frac{2}{3x} - \frac{1}{6x}$.

9. $\frac{2x}{a - 3b} \cdot \frac{6b - 2a}{2xy}$.

10. $\frac{3a}{a + b} - \frac{ab - b^2}{a^2 - b^2}$.

11. $\frac{a + b}{2x} + \frac{2a - 3b}{3x} - \frac{3a - 4b}{4x}$.

12. $\frac{x^2 - y^2}{2x} \div \frac{ax - ay}{6x^3}$.

Simplify the following expressions:

13. $\frac{1}{x - 1} - \frac{7}{1 - x}$.

14. $\frac{a + \frac{b}{c}}{a - \frac{b}{c}}$.

15. $\left(x - \frac{1}{y}\right) \div \left(x + \frac{1}{y}\right)$.

Evaluate each of the following expressions, letting $a = 1$, $b = 2$, and $c = 3$:

16. $\frac{a^2 + 3ab - b^2}{4a + 5b + 6c}$.

17. $\frac{a^2}{b^3} - \frac{b}{c} + \frac{a^2}{bc}$.

18. If a man earns a dollars a week for n weeks, and c dollars a week during the following p weeks, what is his average weekly income during this period of time?

19. If x books cost y dollars, how many books can be bought for z dollars?

20. If $x=5$, which fraction is larger, $\frac{3}{2x^2-5}$ or $\frac{5}{6x+10}$?

165. Summary. In this chapter you have learned

(1) How to change fractions to equivalent fractions by applying the Fundamental Principle of Fractions.

(2) How to find the L. C. D. of two or more fractions.

(3) How to combine fractions by addition, by subtraction, by multiplication, and by division.

(4) How to change the signs of a fraction.

(5) How to simplify complex fractions.

Test on Chapter XII

1. Supply the missing term in each of the following:

$$(1)\ \frac{a+3}{2a}=\frac{(\quad)}{10a};\quad (2)\ \frac{2x^2-8x}{6x^2}=\frac{x-4}{(\quad)};$$

$$(3)\ \frac{4}{a+b}=\frac{(\quad)}{a^2-b^2}.$$

2. Reduce to lowest terms $\frac{2r^2-6rs}{10r^2}$.

3. Find the *sum*, the *difference*, the *product*, and the *quotient* of the two fractions in each case:

$$(1)\ \frac{5}{6n^2},\ \frac{8}{15n^3};\quad (2)\ \frac{5x-15}{3},\ \frac{x-3}{5}.$$

4. Combine $\frac{a}{12}-\frac{2a-4}{8}+\frac{2a+3}{6}$.

5. Simplify $\left(1+\frac{b}{a}\right)\cdot\left(1-\frac{b}{a}\right)\div\left(\frac{a^2-b^2}{a^2}\right)$.

6. Show that $\frac{a+b}{a-b} - \frac{a-b}{a+b}$ equals $\frac{4ab}{a^2-b^2}$.

7. Show that $x = 3\frac{1}{2}$ satisfies the equation $\frac{x+4}{3x-8} = 3$.

8. Indicate which of the following equalities are correct:

(1) $\frac{4(a+b)}{4a+4b} = 1$; (3) $\frac{a}{b} + 1 = a + b$;

(2) $\frac{7-x}{7-y} = \frac{x}{y}$; (4) $\frac{3c+5d}{3x} = \frac{c+5d}{x}$.

9. Given the fraction $\frac{n}{d}$, in which n and d are positive integers greater than 3. In the case of each of the following fractions, indicate whether its value is greater than, less than, or equal to $\frac{n}{d}$:

(1) $\frac{n}{d-3}$; (2) $\frac{n+5}{d}$; (3) $\frac{4n}{4d}$; (4) $\frac{n}{2d}$.

10. If an automobile has covered a distance of m miles in h hours, how much farther can it go in 3 hours at the same rate?

Exercises. *Review*

1. Multiply $3n^2 - n + 5$ by $2n - 7$. Check the result.

2. Divide $17x^2 - 25x + 10x^3 + 4$ by $5x - 4$. Check the result.

3. Solve the equation

$$(3x-8)(3x+8) = 9x(x-2) + 26.$$

4. Solve the equation

$$(4x-5)^2 = 4 - (7-2x)(1+8x).$$

5. Solve graphically and algebraically the pair of equations:

$$2x - y = -8,$$
$$x + 3y = 3.$$

6. Solve the equation $x^2 - 7x = 18$.

7. According to a recent postal regulation, the cost of sending air mail is subject to the rule that "5 cents must be paid for the first ounce and 10 cents for each additional ounce."

(1) State this rule in a short form by means of a formula.

(2) Make a table showing the cost c of sending any number of ounces, n, from 1 to 10.

(3) From this table obtain a formula showing the relation of c and n. Show that this formula agrees with the one obtained in (1).

(4) Using each of these formulas, find the cost of sending a two-pound package by air mail. Which formula is more convenient?

8. (1) Evaluate the formula $C = \frac{5}{9}(F - 32)$ for the following successive values of F: -4, 14, 23, 32, 59, 77. Using these related values of C and F, draw a graph picturing the formula.

(2) From the graph find the value of C when $F = 68$.

9. Find three consecutive whole numbers such that the square of the second number minus the square of the first equals the third number.

10. A man traveled d miles in h hours. At this same rate, how far could he travel in 5 hours?

11. John's average in the first seven algebra tests was a per cent; his average in the following five tests was b per cent. What was his average in all the tests?

12. A school-supply store sold both 2-cent and 5-cent pencils. On one day 23 pencils were sold. If the receipts from these sales were 70 cents, how many pencils of each kind were sold?

CHAPTER XIII

FRACTIONAL EQUATIONS

1. FRACTIONAL EQUATIONS CONTAINING ONE OR TWO UNKNOWNS

166. Fractional Equations. An equation containing a fraction in which the unknown occurs in the denominator is called a *fractional equation.* Equations of this kind frequently arise in the solution of problems.

Illustrative Example. An airplane pilot made a trip of 400 miles. He returned in another plane which traveled 20 miles per hour faster than the first machine, thus requiring one hour less to make the trip. Find the hourly rate of each machine.

Let $r =$ the rate per hour of the first machine.
Then $r + 20 =$ the rate per hour of the second machine.
Hence we obtain the fractional equation

$$\frac{400}{r+20} + 1 = \frac{400}{r}. \qquad \text{(Explain.)}$$

You will learn in this chapter how to solve equations of this kind.

Equations with fractional coefficients, such as $\frac{1}{2}x + x + \frac{5}{6}x = 5$, and $\frac{x+5}{5} - \frac{x}{2} = \frac{3x}{4}$, which were discussed in Chapter IX, will be included in the present chapter, since the method used in solving them also applies in the case of fractional equations.

167. Clearing an Equation of Fractions. An equation containing fractions may be transformed into an equation having no denominators by multiplying both members by an expression that contains each of the denominators among its factors. Evidently the L.C.D. of the given fractions satisfies this condition; hence it is generally used as the multiplier. This process is called *clearing the equation of fractions.* The resulting equation can then be solved in the usual way.

Illustrative Example 1. Solve the equation

$$\frac{n-2}{4}-\frac{n-4}{6}=\frac{2}{3}.$$

Solution. Here the L.C.D. is 12. Multiplying each member by 12, we have

$$\frac{12\cdot(n-2)}{4}-\frac{12\cdot(n-4)}{6}=\frac{12\cdot 2}{3}.$$

Reducing each fraction to its lowest terms, we have

$$3(n-2)-2(n-4)=8.$$

Removing parentheses, we obtain

$$3\,n-6-2\,n+8=8.$$

Hence $$n+2=8,$$

and $$n=6.$$

Check. Substitute 6 for n in the original equation.

Illustrative Example 2. Solve the equation

$$\frac{2\,x+1}{x-3}=\frac{5\,x}{3\,x-9}-\frac{1}{6}.$$

Solution. The L.C.D. is $6(x-3)$. Hence we have

(1)	$\frac{6(x-3)\cdot(2\,x+1)}{x-3}=\frac{6(x-3)\cdot 5\,x}{3(x-3)}-\frac{6(x-3)\cdot 1}{6}.$	$M_{6(x-3)}$.
(2)	$6(2\,x+1)=10\,x-(x-3).$	Why?
(3)	$12\,x+6=10\,x-x+3.$	Rule of Distr.
(4)	$3\,x=-3.$	Tr.
(5)	$x=-1.$	D_3.

Check. Substitute -1 for x in the original equation.

Exercises. *Equations containing Fractions*

Numbers 1 to 6, oral

Solve each of the following equations:

1. $\frac{x}{4}=8.$
2. $\frac{4x}{5}=12.$
3. $\frac{3}{5}n=15.$
4. $\frac{n+3}{2}=1.$
5. $\frac{2y-7}{3}=7.$
6. $\frac{2x}{7}=\frac{3}{5}.$

Solve each of the following equations and check each solution:

7. $\frac{x}{5}+\frac{2x}{5}=3.$
8. $\frac{n}{2}-\frac{n}{3}=5.$
9. $\frac{2a}{3}+\frac{a}{5}=\frac{52}{3}.$
10. $\frac{y}{2}+\frac{y}{3}-\frac{y}{4}=7.$
11. $\frac{x}{2}-\frac{2x}{3}=1+\frac{x}{6}.$
12. $\frac{n}{6}=\frac{n}{2}-10.$
13. $\frac{3a}{8}+5=\frac{a}{6}.$
14. $\frac{1}{5}(n+4)=(\frac{1}{4}n-3).$
15. $\frac{s-8}{4}=\frac{3s}{10}.$
16. $\frac{3c-1}{9}=\frac{c+9}{12}.$
17. $\frac{n-7}{6}+\frac{7-n}{3}=\frac{7}{9}.$
18. $\frac{5x-1}{4}=\frac{1}{14}-\frac{7x-3}{7}.$
19. $\frac{2x-7}{5}-\frac{3x-2}{10}=\frac{1}{5}.$
20. $\frac{3x}{4}=\frac{3x-1}{2}-\frac{2x}{3}.$
21. $\frac{n+4}{5}-\frac{n-3}{3}=3.$
22. $\frac{x+1}{4}+\frac{x+2}{5}=2.$
23. $\frac{y+8}{9}-2=\frac{y-6}{12}.$
24. $\frac{D-6}{14}-\frac{2(6-D)}{21}=-\frac{1}{6}.$
25. $\frac{r+5}{2}-\frac{2r-5}{8}=\frac{2r}{3}.$
26. $\frac{8n-1}{6}=1\frac{1}{2}-\frac{10n+7}{12}.$
27. $\frac{6(3-x)}{5}-\frac{12(1-x)}{25}=3.$
28. $\frac{8(9-2s)}{21}=4\frac{5}{6}-\frac{5(5+2s)}{14}.$
29. $\frac{4-10x}{3}+\frac{40x-5}{4}=\frac{3}{4}.$

30. If 99 is subtracted from a certain number, the result is the same as that obtained when the given number is divided by 10. What is the number?

31. If a certain number is subtracted from 75, the result is half the number. What is the number?

32. One eighth of a certain number is 3 less than one sixth of that number. What is the number?

33. What number, when added to $6\frac{1}{4}$, gives the same result as when the number is multiplied by $7\frac{1}{4}$?

34. The numerator of a certain fraction is 7 smaller than the denominator. If 3 is added to the numerator, the value of the resulting fraction is $\frac{3}{4}$. Find the original fraction.

35. A man left one third of his property to his son, one fourth to his daughter, and the rest, which was $15,000, to his wife. How much did each of the children receive?

168. Equations containing Decimals. Decimals represent fractions whose denominators are 10, 100, 1000, and so on. Hence equations which contain decimals may readily be "cleared of fractions" by using a multiplier containing a power of 10 that corresponds to the largest number of decimal places appearing in any term.

Exercises. *Equations containing Decimals*

Solve each of the following equations:

1. $.4\,x = .8$.
2. $.06\,c = 1.44$.
3. $.5\,x = .2\,x + 9$.
4. $.06\,n - 8 = .04\,n$.
5. $.3\,s + .45 = 1.5 - .05\,s$.
6. $7 + .03\,b = .06\,b + 1.6$.
7. $.8(2\,x - 1) = 2.4$.
8. $.04(s + 50) = 5$.
9. $4.5 = .25(30 - x)$.
10. $.06(900 - n) + .04\,n = 19.5$.

11. $.8 + x = .20(9 + x)$.

12. $.10(24 - a) = 2.8$.

13. $6 + .045\,c = 10.5$.

14. $.06(500 - c) - .08\,c = 7.6$.

15. $.035\,p + .05(3000 - p) = 132$.

16. $\dfrac{.7\,x - 1}{3} = .12\,x - .05$.

17. $\dfrac{1.5\,x - 6}{5} = \dfrac{3.25\,x - 12}{11}$.

18. $\dfrac{.2\,x + 3}{10} + .7\,x - 1 = \dfrac{1.4\,x}{2}$.

19. Twenty-eight coins composed of quarters and dimes amount to $4. How many coins of each kind are there?

20. An ice-cream stand sold both 5-cent and 10-cent cones. On a certain day the receipts were $65 from the sale of 970 cones. How many of each kind were sold?

Exercises. *Fractional Equations*

Solve each of the following equations and check the results:

1. $\dfrac{4}{n} + \dfrac{2}{n} = 3$.

2. $\dfrac{4}{x} - \dfrac{3}{x} + \dfrac{5}{x} = \dfrac{2}{3}$.

3. $\dfrac{3}{x} - \dfrac{4}{3\,x} = \dfrac{5}{3}$.

4. $\dfrac{1}{4} = \dfrac{1}{d} + \dfrac{1}{8}$.

5. $\dfrac{1}{n} - \dfrac{1}{3\,n} + \dfrac{1}{6\,n} = \dfrac{1}{6}$.

6. $\dfrac{3}{5} + \dfrac{3}{n} = 1$.

7. $\dfrac{100}{r} - 4 = \dfrac{200}{5\,r}$.

8. $\dfrac{d + 1}{d} = \dfrac{1}{2}$.

9. $\dfrac{r - 8}{2\,r} = \dfrac{r + 8}{r} - \dfrac{3}{4}$.

10. $\dfrac{x + 3}{2\,x} - \dfrac{5}{6} = \dfrac{2 - x}{x}$.

11. $\dfrac{1}{m} - \dfrac{m - 1}{3\,m} = \dfrac{2}{m}$.

12. $\dfrac{2\,t + 1}{3\,t} - \dfrac{2}{3} = \dfrac{1}{9}$.

13. $\frac{2}{3}-\frac{7-2x}{3x}=\frac{3}{4}.$

14. $\frac{4}{5}-\frac{3n-4}{7n}=\frac{1}{n}+\frac{2}{7}.$

15. $\frac{3y+7}{2y}-\frac{2y-1}{3y}=1\frac{5}{16}.$

16. $\frac{d-2}{d+1}=\frac{1}{3}.$

17. $\frac{16}{20+q}=\frac{70}{100}.$

18. $\frac{12+n}{29+n}=\frac{1}{2}.$

19. $\frac{6}{100-a}=\frac{1}{10}.$

20. $\frac{5}{40+n}=\frac{1}{25}.$

21. $\frac{20}{100+w}=\frac{18}{100}.$

22. $\frac{1.6}{20+p}=\frac{1}{20}.$

23. $\frac{175}{n}=\frac{217}{n+12}.$

24. $\frac{1}{d-1}+\frac{1}{3}=\frac{8}{15}.$

25. $\frac{60}{n}+2=\frac{60}{n-5}.$

26. $\frac{1}{6+n}+\frac{2}{3(6+n)}=-\frac{5}{9}.$

27. $\frac{1}{6x+1}-\frac{3x-3}{12x+2}=\frac{2}{3}.$

28. $\frac{300}{x}-\frac{300}{x+2}=25.$

29. $\frac{3}{4x+5}=\frac{2}{3}-\frac{5}{12x+15}.$

30. $\frac{400}{g}-2=\frac{400}{g+10}.$

31. $\frac{8}{x+3}=\frac{3}{x-3}.$

32. $\frac{20}{x+2}=\frac{3}{5}\left(\frac{20}{x-2}\right).$

33. $\frac{9}{x-5}+\frac{4}{x+5}=\frac{3x+40}{x^2-25}.$

34. $\frac{480}{r+16}+1=\frac{480}{r}.$

35. $\frac{x-1}{2x-1}-\frac{4}{4x^2-1}=\frac{x+1}{2x+1}.$

36. $\frac{8}{10+r}+\frac{8}{10-r}=1\frac{2}{3}.$

37. $\frac{4}{3x+9}-\frac{5}{4x-12}=\frac{15}{2x^2-18}.$

38. $\frac{x-1}{x-3}+\frac{2x-3}{2x-6}-\frac{4x-3}{4x-12}=\frac{1}{2}.$

39. The denominator of a fraction is 5 larger than the numerator. If 1 is subtracted from the numerator, the value of the resulting fraction is $\frac{1}{4}$. Find the original fraction.

40. What number must be added to the numerator and the denominator of the fraction $\frac{29}{69}$ to make the resulting fraction equal to $\frac{1}{2}$?

41. What number must be subtracted from each term of the fraction $\frac{15}{23}$ so that the resulting fraction shall equal $\frac{5}{9}$?

42. What is the number which, if added to the numerator and subtracted from the denominator of the fraction $\frac{7}{8}$, will cause the resulting fraction to have the value $1\frac{1}{2}$?

43. What number must be added to the numerator and the denominator of the fraction $\frac{5}{8}$ so as to double the value of the fraction?

169. Simultaneous Equations containing Fractions. Pairs of equations in which either or both of the equations contain fractions may be cleared of fractions as explained above. The resulting equations, which will involve only integral coefficients, may then be solved by any of the methods explained in Chapter X.

Exercises. *Pairs of Equations containing Fractions*

Solve the following pairs of equations:

1. $\frac{x}{2}+\frac{y}{3}=4,$

$\frac{2x}{3}-\frac{y}{2}=-\frac{1}{3}.$

2. $\frac{a}{6}-\frac{3b}{2}=-4,$

$\frac{2a}{3}-\frac{b}{3}=1.$

3. $\frac{3R}{5}+\frac{3D}{2}=9,$

$\frac{R}{2}-\frac{5D}{4}=2\frac{1}{2}.$

4. $2x+\frac{5y}{3}=6,$

$x-\frac{y}{4}=3.$

5. $\frac{n+1}{d}=\frac{1}{2}$,
$\frac{n}{d-4}=\frac{3}{4}$.

6. $\frac{88}{r+c}=4$,
$\frac{90}{r-c}=5$.

7. $\frac{400}{s+w}=5$,
$\frac{400}{s-w}=6\frac{2}{3}$.

8. $x-y=50$,
$.3\,x-.5\,y=19$.

9. $.05\,x+.04\,y=108$,
$x+y=2500$.

10. $d+q=50$,
$.1\,d+.25\,q=9.8$.

11. $c+d=7000$,
$.04\,c-.06\,d=30$.

12. $a+c=234$,
$.25\,a+.15\,c=53.10$.

13. $.02\,x+.01\,y=1.4$,
$x+y=90$.

14. $.1\,d+.05\,n=1.65$,
$d+n=18$.

15. $\frac{1}{x}+\frac{1}{y}=\frac{5}{6}$,
$\frac{1}{x}-\frac{1}{y}=\frac{1}{6}$.

SUGGESTION. We may first regard $\frac{1}{x}$ and $\frac{1}{y}$ as the unknowns. It is easy to eliminate either $\frac{1}{x}$ or $\frac{1}{y}$ by addition or subtraction. Then we can complete the solution in the usual way.

16. $\frac{2}{x}-\frac{9}{y}=2$,
$\frac{1}{x}-\frac{3}{y}=\frac{1}{2}$.

17. $\frac{3}{x}+\frac{3}{y}=\frac{5}{8}$,
$\frac{9}{x}+\frac{2}{y}=1$.

18. $\frac{8}{x}+\frac{3}{y}=1$,
$\frac{4}{x}-\frac{6}{y}=1$.

2. PROBLEMS

170. Work Problems. In industrial and mechanical work it is often necessary to find the amount of time needed to complete a certain piece of work, especially when several persons or machines are involved. Such problems are usually called *work problems*. A few of the simplest types of these problems will be introduced in the following exercises.

Illustrative Example. Suppose that a workman requires 5 hours to do a piece of work which a second man can do in 3 hours. How many hours will it take to complete the work if both men work together?

Solution. Let $h =$ the number of hours required when both men work together. The conditions of the problem may be tabulated as follows:

	FIRST MAN	SECOND MAN
Hours required	5	3
Work per hour	$\frac{1}{5}$	$\frac{1}{3}$
Hours worked	h	h
Total work done	$\frac{h}{5}$	$\frac{h}{3}$

Since the two men together can do $\frac{1}{h}$ of the work in 1 hour, the total work they will do in h hours is $\frac{1}{h} \cdot h$, or 1. Hence we have the following equation:

$$\frac{h}{5} + \frac{h}{3} = 1.$$

Solving this equation, we find that $h = 1\frac{7}{8}$; that is, $1\frac{7}{8}$ hours will be required to complete the work if both men work together.

Exercises. *Work Problems*

1. Ralph can mow the lawn in 30 minutes, but his younger brother Paul takes 50 minutes for the same task. If the two boys work together, each using a separate lawn mower, how long will it take them to mow the lawn?

2. A daily newspaper has two presses, one of which can print the entire edition in $1\frac{1}{2}$ hours, while the second would require 2 hours to do the same work. If both presses are used, how many hours will it take to print the edition?

3. B can do a piece of work in 10 days, and C can do it in 6 days. In how many days can they do it when working together?

4. Mr. Brown can lay a certain cement walk in 16 hours. Mr. Stout can do the same piece of work in 12 hours. If Mr. Stout hires Mr. Brown to help him, how long will it take them together to lay the walk?

5. A gasoline tank can be filled by one pipe in 10 minutes and by another in 14 minutes. How long will it take to fill the tank if both pipes are used at the same time?

6. A wash bowl can be filled in 35 seconds if the cold-water faucet is turned on, or in 30 seconds if the hot-water faucet is used. How long will it take to fill it if both faucets are used?

7. How long will it be before the bowl in Exercise 6 will overflow if the drain is open and both faucets are turned on, provided the drain can empty the bowl in 25 seconds?

8. A piece of work was started on a machine which could finish it in 6 hours. After 4 hours the machine broke down, and the job had to be finished on a smaller machine which would have taken 8 hours for the entire job. How long did it take the second machine to complete the work?

9. Joe can wash his father's car in 90 minutes, while Harold can do it in 50 minutes. After Joe has worked for 20 minutes, Harold joins him and together they finish the task. How long does Harold work?

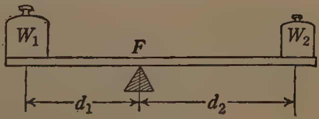
10. A garage mechanic gave his helper the job of overhauling a car, which he figured would take the helper 10 hours. After the helper had been working for 3 hours, the owner of the car called and asked to have the job "rushed through." The mechanic, who could have done

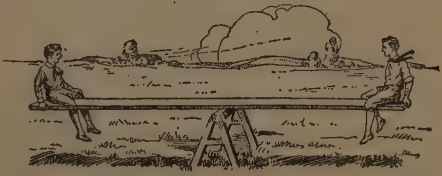
the whole task in 6 hours, then joined his helper and together they finished the job. How long was it after the owner called before his car was ready?

11. In a factory a workman started a grinding job which would take him 5 hours to complete. After he had worked 4 hours, he was taken off the job and given another assignment. A second workman took his place and the job was finished in $1\frac{1}{2}$ hours more. How long would it have taken the second man to do the whole job?

12. Two typists were asked to duplicate a long report. It was estimated that the first girl would take 6 hours to do the work alone, while the second girl would require 8 hours. After they had worked together for $2\frac{1}{2}$ hours, the first typist was given another assignment, and the second finished the work alone. How long did it take the second typist to complete the task?

171. Lever Problems. When two children play on a teeter board, they are really illustrating an important relation which is known as the *law of the lever*. The word *lever* refers to a rod or a board which is supported or balanced at some point, as in the figure. The point of support, F, is called the *fulcrum*.





The teeter board represents one form of lever. Suppose that the weights of two boys playing on a teeter board are W_1 and W_2, respectively, and that their distances from the fulcrum are d_1 and

d_2 respectively. It can be shown that these weights will "balance" whenever $W_1d_1 = W_2d_2$, no account being taken of the weight of the lever.

This formula, known as the *law of the lever*, can also be written in the fractional form $\frac{W_1}{W_2} = \frac{d_2}{d_1}$.

Thus, if $W_1 = 70$ pounds, $d_1 = 6$ feet, and $d_2 = 5$ feet, we have $5\,W_2 = 6 \times 70$; that is, $W_2 = 84$ pounds.

NOTE. In formula work frequent use is made of "subscripts" to indicate different quantities of the same kind. Thus, in the above formula, W_1 and W_2 are used to indicate two different weights. W_1 is read "W sub 1," W_2 is read "W sub 2," and so on. It is necessary to distinguish carefully between *exponents* and *subscripts*. For example, d^2 means d *square*, while d_2 indicates a particular distance, called "d sub 2."

Exercises. *Lever Problems*

1. State in words the meaning of the formula $W_1d_1 = W_2d_2$.

In each of the following four problems it is assumed that the lever is supported at its center.

2. A boy weighing 56 pounds and his sister, who weighs 40 pounds, are playing on a teeter board. If the girl sits 7 feet from the fulcrum, how far from it must the boy sit to balance the weight of his sister?

3. Robert and Arthur are playing seesaw. Their weights balance when Robert is 6 feet from the fulcrum and Arthur is 7 feet from the fulcrum. If Robert weighs 84 pounds, how much does Arthur weigh?

4. Edward and Roy have a 14-foot plank with which they wish to play seesaw. Edward weighs 96 pounds and Roy 72 pounds. If Roy is seated at one end of the plank, how far from the fulcrum must Edward sit if the weights of the boys are to balance?

5. Tom and Dick balance on a teeter board when they are 6 feet and 8 feet, respectively, from the point of support. If, however, Dick is replaced by Harry, who weighs 5 pounds more, it is found that Tom must sit at a distance of $6\frac{1}{2}$ feet from the fulcrum. What is the weight of each boy?

6. A rock weighing 200 pounds is to be raised by using a crowbar 5 feet long. If the support (fulcrum) is placed 1 foot from the end of the crowbar upon which the rock rests, how much pressure must be exerted at the other end of the bar to raise the rock? If a bar 6 feet long were used, how much pressure would be needed? Which bar requires less effort? (The weight of the crowbar is not to be taken into account.)

Exercises. *Miscellaneous Problems*

1. The sum of two numbers is 28, and the first number is three fourths as large as the second. What are the numbers?

2. Find two numbers whose sum is 47 and whose quotient is $8\frac{2}{5}$.

3. The difference between two numbers is 15. If their sum is divided by the smaller number, the resulting quotient is equal to $\frac{13}{4}$. Find the numbers.

4. A shoe dealer sold a pair of shoes for $8, thus making a profit of 25%. How much did the shoes cost him?

5. A merchant found that his "overhead expense" in handling certain goods amounted to 10% of the cost of the goods. To clear expenses and also to make a profit of 20% of the cost, he had to sell the goods for $260. What was the original cost?

6. A man bought a new car for which he paid $465 cash. The amount allowed on the old car was $25 less than one fifth of the value of the new one. What was the cost of the new car, and how much was allowed for the old car?

7. Mr. Smith's house and furniture, which together were insured for $10,000, were partly destroyed by fire. The insurance company allowed him one quarter of the insured value of the house and three quarters of the insured value of the furniture. The amount received for the house exceeded the amount received for the furniture by $500. What was the insured value of the house?

8. The tickets for a football game were sold at a price of $.75 for reserved seats and of $.50 for unreserved seats. The receipts from the sale of 475 tickets were $287.50. How many tickets of each kind were sold?

9. The gate receipts at a baseball game were $1845, from 5380 paid admissions. If grandstand tickets sold at $.50 and all other tickets at $.25, how many of each kind were sold?

10. The members of the senior class of a certain high school were to be assessed equally to buy a class memorial costing $54. Six pupils dropped out of school, and as a result the assessment for those remaining had to be raised 10 cents each. How many members did the class have originally?

11. A group of persons, desiring to buy a birthday gift for a friend, agreed to contribute equally toward it. The gift was to cost $15. Four other friends who heard of the plan asked the privilege of contributing also. This reduced the sum contributed by each person by $1. How many persons actually contributed toward the gift, and how much did each pay?

INVESTMENT PROBLEMS

12. A sum of \$2000 is invested in stocks and bonds. The stocks yield 7% interest and the bonds 4%. If the annual income from the two investments is \$95, how much money is invested in stocks and how much in bonds?

SUGGESTION. Review the method used in solving percentage problems, as outlined in § 123.

13. Mr. Long receives an annual income of \$258 from two investments which total \$4800. A part of the money pays an annual dividend of 6%, and the remainder pays 5%. How much money is invested at each rate?

14. Mr. Jones made an investment in 7% stock. He also invested four times as much money in 5% mortgages. The income from the two investments was \$540 a year. How much money was invested in mortgages?

15. A lawyer invested \$3000, a part at 5% and the remainder at 4%. The annual income from both investments was \$128. How much money was invested at each rate?

16. Miss West invested \$3200 in stocks and bonds. The stocks yielded 8%, and the bonds 4%. If the yearly income from the bonds was \$44 more than the income from the stocks, how much money was invested in stocks? in bonds?

17. Mrs. Davis has invested \$1500 at 3%. How much money must she invest at 8% so as to give her a return of 6% on both investments?

18. By investing two sums of money, namely, \$12,000 and \$8000, Mr. Adams obtained a yearly income of \$1040. If the rate of interest on the larger investment was 2% larger than that on the other, find the two rates of interest.

19. Mr. Hall invested \$7000 in two business enterprises. On one he gained 12% and on the other he lost 3%. If his net gain was \$465, how much money was invested in each enterprise?

MOTION PROBLEMS

20. If the speed of an automobile is increased by 5 miles an hour, it will be able to go 150 miles in the same time which it took to cover 120 miles at the slower rate. Find the original rate.

21. A man traveled 75 miles in an automobile in $2\frac{1}{4}$ hours. If he traveled at the rate of 20 miles an hour when passing through towns, and at the rate of 35 miles an hour in the country, how much time was required to drive through the towns?

22. The senior class of a school went on a trip, covering 270 miles in 8 hours, using first a train and then a boat. The train averaged 45 miles an hour, and the boat averaged 15 miles an hour. How many miles of the trip were covered by train?

23. A motor boat can go upstream 22 miles in 2 hours, while it can make the return trip in 32 minutes less. Find the rate of the boat and the rate of the current.

24. A steamer whose rate in still water is 12 miles an hour can go 60 miles down a river in the same time that it can go 36 miles upstream. What is the rate of the current?

25. An air-mail pilot made a trip of 300 miles. On the return trip he increased his speed by 25 miles an hour and took one hour less. What was his speed on the return trip?

26. An automobile made a trip of 240 miles. If it had averaged 5 miles an hour more, it could have gone 40 miles farther in the same time. What was its average speed?

27. On a river whose current flows at the rate of 2 miles per hour, a motor boat can go 24 miles downstream in the same time that it requires to go 16 miles upstream. Find the rate of the boat.

MIXTURE PROBLEMS

28. Walnuts which sell for 35 cents a pound are to be mixed with Brazil nuts which sell for 40 cents a pound. How many pounds of each must be used to obtain 80 pounds of mixed nuts which can be sold for 38 cents a pound?

29. A confectioner wishes to mix candy which sells for 40 cents a pound with a grade that sells for 70 cents a pound, so that he can sell the mixture for 59 cents a pound. If he wants to make up 60 one-pound boxes, how many pounds of each kind should be used?

30. A baking company wished to make up 600 one-pound boxes of assorted cakes to sell for 30 cents a box. Cakes were to be used which retailed at 28 cents and 34 cents a pound. How many pounds of each kind should be used?

31. A grocer has 20 gallons of vinegar which is 80% pure. How much water must be added to reduce it to a 60% mixture?

32. A dealer has 14 pounds of nitric acid 70% pure. How many pounds of water must he add to obtain a mixture 22% pure?

33. How many pounds of water must be added to 4 pounds of hydrochloric acid which is 37% pure in order that the resulting mixture may be only 19% pure?

34. If the radiator of an automobile contains 14 quarts of a solution which is 12% alcohol, how much alcohol must be added to make it a 20% solution?

35. How much water must be evaporated from 12 gallons of a 10 % salt solution so that the remainder will be a 15 % solution?

3. Literal Equations and Formulas

172. Literal Equations containing One Unknown. When the conditions of a verbal problem are given in *numerical* form, the equation by which we attempt to solve such a problem is said to be a *numerical equation.* The solution obtained from such an equation obviously fits only the *particular* numerical situation described in the problem. If, however, the conditions of a verbal problem are given in a *general* form, — that is, if letters are used instead of numbers, — the resulting equation is a *literal equation.* In that case the solution represents a *formula* by which we may solve *any* problem of the given type.

The solution of general verbal problems is thus seen to depend very largely on one's skill in solving *literal equations,* simple cases of which were considered in Chapter IX. It is now possible to deal with general problems and literal equations of a more complicated type.

Unless there is a remark to the contrary, it is customary to regard letters taken from the latter part of the alphabet as the unknowns to be found.

Illustrative Example. Solve the equation

$$\frac{x}{b}+\frac{x}{a}=a+b.$$

Solution.

$$\begin{array}{lr|l} (1) & ax+bx=a^2b+ab^2. & M_{ab}. \\ (2) & x(a+b)=ab(a+b). & \text{Factoring.} \\ (3) & x=ab. & D_{a+b}. \end{array}$$

Check.

$$\frac{a\cancel{b}}{\cancel{b}}+\frac{\cancel{a}b}{\cancel{a}}=a+b,$$

or

$$a+b=a+b.$$

Exercises. *Literal Equations*

Solve each of the following equations for x:

1. $\frac{2x}{3} = b.$
2. $\frac{a}{x} = 3.$
3. $\frac{c}{5} = \frac{2}{x}.$
4. $h = \frac{5x}{2.5}.$
5. $n = \frac{2lx}{3n}.$
6. $\frac{r}{s} = \frac{4}{x}.$
7. $\frac{x-a}{10} = 2a.$
8. $\frac{x-n}{n} = a.$
9. $\frac{x}{8} = \frac{x-b}{6}.$
10. $\frac{x}{m} - 1 = \frac{x}{2m}.$
11. $\frac{x-a}{2a} = \frac{1}{6}.$
12. $\frac{x-d}{d} = s + \frac{r}{d}.$
13. $\frac{x}{a} - \frac{x}{b} = k.$
14. $\frac{1}{p} + \frac{1}{q} = \frac{1}{x}.$
15. $x + \frac{rx}{100} = a.$
16. $.01\,rw = .01\,p(w + x).$
17. $\frac{ax}{100} + \frac{b(x-p)}{100} = r.$
18. $\frac{ra}{100} + x = \frac{s(a+x)}{100}.$
19. $\frac{3m}{2x - m} = 5.$
20. $\frac{c+x}{d+x} = 5.$
21. $\frac{d}{x} = \frac{d+30}{x+5}.$
22. $\frac{a-x}{b-x} = \frac{b}{a}.$
23. $\frac{m}{x+c} = \frac{m-3}{x-c}.$
24. $\frac{d}{r-x} = \frac{d+10}{r+x}.$

25. If the number x is added to the numerator and the denominator of the fraction $\frac{n_1}{d_1}$, the value of the resulting fraction is $\frac{n_2}{d_2}$. Find x.

26. What number must be subtracted from the numerator and the denominator of the fraction $\frac{a}{b}$ in order that the value of the resulting fraction may become $\frac{c}{d}$?

27. A certain sum of money (p) is invested in two parts, one part (x) at r % and the other part at s %. The annual

income from the two investments is i. How much is invested at each rate?

28. An airplane covered a distance of d miles at the rate of r miles an hour. How much must the speed per hour be increased on the return trip if the pilot wishes to improve his record by 2 hours? by n hours?

29. One man can do a piece of work in d_1 days, and a second man can do it in d_2 days. How long will it take them when they work together?

30. One man can do a piece of work in h_1 hours, and a second man can do it in h_2 hours. The first man works alone for n hours and then the second man joins him. How long will it take them to complete the work?

31. How much pressure (p) must be exerted on the end of a crowbar in order to raise an object weighing w pounds if the crowbar is l feet long and the fulcrum is d feet from the object to be moved?

32. How many quarts of water (q) must be added to n quarts of a solution containing $r\%$ of some chemical so that the new solution will be a $p\%$ solution?

Exercises. *Pairs of Literal Equations*

Solve the following pairs of equations:

1. $x + y = a,$
   $x - y = b.$

2. $ax + 3y = n,$
   $2ax + y = s.$

3. $3x + 5y = c,$
   $2x + 3y = d.$

4. $nx - ny = d,$
   $nx + ny = s.$

5. $x + y = 2,$
   $cx - dy = 2.$

6. $2bx - 4cy = b,$
   $5bx + 3cy = c.$

7. $y = ax,$
$x - y = d.$

8. $\frac{x}{a} + \frac{y}{b} = a + b,$
$x + y = ab.$

9. $x_1 + x_2 = s,$
$.05\,x_1 + .04\,x_2 = i.$

10. $y_1 + y_2 = t,$
$.05\,y_1 - .06\,y_2 = d.$

11. The sum of two numbers is s, and their difference is d. Find the numbers.

12. Two sums of money, s_1 and s_2, amounting to a dollars, are invested at $r\%$ and $p\%$ respectively. The annual income from the two investments is i dollars. Find s_1 and s_2 in terms of the other letters used.

173. Transformation of Formulas. Whenever we solve a literal equation for a specified letter, we are really *transforming* the equation by applying certain fundamental principles which have been considered in the preceding chapters. In like manner, we may solve a given formula for any letter appearing in the formula. The exercises which follow will serve to furnish additional practice in transforming formulas that are used in various branches of applied mathematics.

Practice Exercises. *Transforming Formulas*

Transform, as indicated, each of the following formulas:

1. Solve $V = Bh$ for h; for B.

2. Solve $E = \frac{W}{I}$ for I; for W.

3. Solve $\frac{R}{E} = \frac{A}{a}$ for E; for a.

4. Solve $\frac{W_1}{W_2} = \frac{d_2}{d_1}$ for W_2; for d_1.

5. Solve $d = \frac{l^2 w}{8\,s}$ for w; for s.

6. Solve $s = \frac{1}{2} at^2$ for a.

7. Solve $W = T_1V - T_2V$ for T_1; for V.

8. Solve $A = b - \frac{br}{100}$ for r; for b.

9. Solve $v = u + \frac{1}{2} at$ for u; for t.

10. Solve $\frac{1}{d_1} + \frac{1}{d_2} = \frac{1}{f}$ for f; for d_2.

11. Solve $I = \frac{E}{R + r}$ for R; for r.

12. Solve $h = \frac{wv^2}{1.25\,r}$ for r; for w.

13. Solve $S = \frac{rl - a}{r - 1}$ for a; for r.

14. Solve $n = \frac{s}{w} - 1$ for s; for w.

15. Solve $L = \frac{Wh}{d(W + P)}$ for d; for W.

Transform and evaluate, as indicated, each of the following formulas:

16. $M = \frac{W}{E}$. (1) Solve for E. (2) Find E if $M = 2.5$ and $W = 37.5$.

17. $\frac{V_1}{V_2} = \frac{P_2}{P_1}$. (1) Solve for V_2. (2) Find V_2 if $V_1 = 8$, $P_1 = 30$, and $P_2 = 60$.

18. $H = \frac{Nd^2}{2.5}$. (1) Solve for N. (2) Find N if $H = 22.5$ and $d = 3\frac{3}{4}$.

19. $S = \frac{W_1}{W_1 - W_2}$. (1) Solve for W_2. (2) Find W_2 if $S = 8.1$ and $W_1 = .365$.

20. $S = \frac{n}{2}(a + l)$. (1) Solve for n. (2) Find n if $S = 77$, $a = 2$, and $l = 20$.

174. Summary. In this chapter you have learned

(1) How to solve fractional equations containing one or two unknowns.

(2) How to solve problems in which fractional equations are necessary.

(3) How to solve a literal fractional equation for any letter appearing in the equation in terms of the other letters.

(4) How to transform formulas containing fractions.

Test on Chapter XIII

What multiplier should be used to clear each of the following equations of fractions?

1. $\frac{x}{3}+\frac{5x}{6}=\frac{7}{8}$.

2. $\frac{x^2}{3x+6}=\frac{x}{3}+\frac{1}{6}$.

Clear each of the following equations of fractions:

3. $\frac{x+4}{2}=\frac{13}{5}-\frac{x-9}{10}$.

4. $\frac{3}{x}+\frac{5}{4x}=\frac{7}{8}$.

Solve each of the following equations:

5. $\frac{x}{2}-\frac{2}{3}=\frac{x}{6}$.

6. $.3x-4.8=.06x$.

7. $\frac{7}{y}-\frac{3y+4}{2y}=\frac{1}{6}$.

8. $\frac{120}{x}-4=\frac{120}{x+5}$.

9. Determine by substitution whether or not $\frac{1}{2}$ satisfies the equation $\frac{3x}{5}+\frac{x}{10}=\frac{7}{20}$.

10. Solve the equation $\frac{x}{4d}+\frac{x}{d}=1$ for x.

11. Solve the formula $T=w+\frac{wa}{g}$ for a.

12. Solve the formula $H=\frac{fs}{f+s}$ for s.

13. A plumber can install a water heater in 6 hours. His helper would require 10 hours to do the same job. How many hours would be required if both men worked together?

14. A man invested \$3000, a part in stocks at 7% and the remainder in bonds at 4%. He received annually \$21 more income from the bonds than from the stocks. How much money did he invest at each rate?

15. Solve the following pair of equations:

$$\frac{a-1}{b}=\frac{1}{2},$$
$$\frac{a}{b+5}=\frac{1}{3}.$$

Exercises. *Review*

1. What principle is used in clearing an equation of fractions?

2. Divide $42-25n^2+12n^3-17n$ by $4n-7$ and check the result.

3. Factor completely each of the following expressions:

(1) $12x^2-23x-9$. (4) $5s^3-10s$.
(2) $25-D^2$. (5) $16x^2+40x+25$.
(3) $3x^2-15x+18$. (6) $2c^4-32d^2$.

4. If $A=\frac{18n+6}{3n}$, and $B=\frac{2n}{3n+1}$, find

(1) $A+B$. (3) $A\times B$.
(2) $A-B$. (4) $A\div B$.

5. If $d=\frac{3}{4}$, find the value of $\frac{d+6}{d}$.

6. If $a=4$, and $b=-3$, find the value of

(1) $\frac{3a}{2b}$. (3) $3a\cdot 2b^3$.
(2) ab^2. (4) a^2-b^3.

7. Solve the equation

$$3-(2x-3)^2=(5-2x)(5+2x)-7$$

and check the result.

8. Solve the equation $x^2+2x=120$ and check the roots.

9. Solve graphically the following pair of equations and check by substitution:

$$2x+y=3,$$
$$x-2y=9.$$

10. In the following table study the relation between the values of s and r:

$s =$	12	30	36		72
$r =$	5	8	9	12	

(1) Write a formula which shows the relation of r and s.

(2) Using this formula, find the missing numbers in the above table.

11. If $\frac{a}{b}$ is a proper fraction in which a and b are positive integers greater than 6, state how each of the following fractions compares in size with $\frac{a}{b}$:

(1) $\frac{3a}{b}$. (2) $\frac{a}{b-2}$. (3) $\frac{a-6}{b+6}$. (4) $\frac{ma}{mb}$. (5) $\frac{a^2}{b^2}$.

12. If a farmer has mowed a acres of grain in h hours, how many more acres can he mow if he works 3 hours longer at the same rate? n hours longer?

13. If a printing press can turn out n newspapers in c minutes, how many hours will it require to print the complete edition of x copies?

CHAPTER XIV

HOW QUANTITIES CHANGE TOGETHER

1. DEPENDENCE; NUMERICAL RELATIONSHIPS

175. Related Changes; Dependence. It was pointed out in a preceding chapter that the world in which we live is a world of *changes*, or fluctuations (see page 166). Many of these changes are closely related. A study of such related changes is very important. For example, we all know that an increase or decrease in the cost of food, clothing, and other necessaries of life affects the welfare of millions of people. Every intelligent citizen follows the "ups and downs" in the prosperity of the community, the state, and the nation.

In this chapter we shall continue our study of quantities that change together. Changing quantities, such as available supplies, prices, and the like, are often called *variables*. *When two variables are so related that a change in the value of one causes a change in the value of the other, we say that one depends upon the other.* A quantity which does *not* change in a given situation is said to be *constant*.

For example, the amount of money which an automobile owner pays out per week for gasoline *depends upon* the price of gasoline per gallon, and also upon the number of gallons bought. If the price per gallon fluctuates, the amount he has to pay for a definite number of gallons also varies.

The number of quarts in a gallon does *not* vary, but is *constant*. Also, the number of feet in a mile is constant, the number of hours in a day is constant, and so on.

Exercises. *Related Changes; Dependence*

In statements 1 to 9 point out the quantities that change together.

1. The *time* it takes a boy to walk to school depends on the *rate* at which he walks.

2. The amount of money saved annually by a family depends on the yearly income and on the amount spent.

3. The number of people that can be seated in an auditorium depends upon the floor space.

4. The amount of coal necessary to heat a building depends upon the size of the building.

5. The premium to be paid on a life-insurance policy of $2000 changes with the age of the insured person at the time the policy is issued.

6. The price of a manufactured article is closely related to the quantity produced.

7. The amount of linoleum needed for a floor depends on the size of the room.

8. The amount of water that can be carried by a pipe depends on the diameter of the pipe.

9. The number of representatives a state has in Congress depends on the population of the state.

10. In what way does the old saying "We get out of a thing what we put into it" illustrate the idea of relationships?

11. In case of a good harvest, are the prices of agricultural products likely to increase or decrease?

12. Whenever there is a scarcity of necessary supplies, are prices likely to go up or to go down?

13. In each of the following expressions, state whether y becomes larger or smaller as x increases:

(1) $y = 2x$. (3) $y = 10 - x$. (5) $y = \frac{5}{4x + 2}$.

(2) $y = x^2$. (4) $y = \frac{1}{x}$. (6) $y = x - \frac{1}{x}$.

Complete each of the following statements:

14. The perimeter of a square depends on the length of -------------.

15. The area of a circle depends on the length of --------.

16. The volume of a cube depends on the length of ------.

17. The numerical value of any algebraic expression depends on the values substituted for -----------------.

18. The number of feet in a yard does not change, but is ----------.

176. Important Methods of Expressing Relationships. It is one of the chief purposes of science and mathematics to study related changes and to find out the laws that govern them. As soon as these laws are clearly understood, we can deal more effectively with situations that previously seemed to be beyond our control. The principal mathematical tools for expressing or representing numerical relationships and for studying dependence are (1) the *formula,* (2) the *table,* (3) the *equation,* and (4) the *graph.*

In the preceding chapters we gave considerable attention to each of these *four* methods of dealing with numerical relationships. We shall review and extend our knowledge of these methods in the exercises that follow.

Exercises. *Expressing Relationships*

1. Point out the related variables in the following statement: "If a train travels at the rate of 40 miles per hour, the distance which it covers depends upon the number of hours which it travels."

2. Show that the relation stated in Exercise 1 can be expressed by means of the formula $d = 40\ h$.

3. The following table is based on the formula $d = 40\ h$. Fill in the missing values.

$h =$	1	2	3	4	5	
$d =$	40	80	120			280

4. Using the table given above, draw a graph of the formula $d = 40\ h$.

5. If d means "distance in miles" and h means "number of hours," translate into words the formula $d = 20\ h$.

6. If c means "cost in cents" and p means "number of pounds," translate into words the formula $c = 10\ p$.

7. If a carpenter earns $12 a day, write a formula expressing his wages (w) for n days.

8. If a newspaper costs 3 cents a copy, write a formula for the number of copies (n) that can be bought for x dollars.

9. To send a telegram, Mr. Smith had to pay 45 cents for the first ten words and 3 cents for each additional word. Write a formula for the cost (c) of a telegram of n words.

10. To have a program printed, I had to pay $4 for the first 100 copies and 50 cents for each additional 100 copies. Write a formula for the total cost (c) of n programs.

11. Make a table based on the formula $p = 5w$, using these successive values of w: 1, 4, 5, 10, 20, 50, 100.

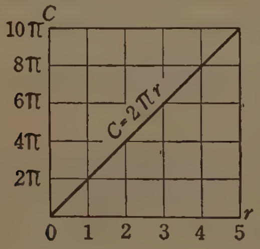
12. Draw a graph of the formula $A = 1.4b$.

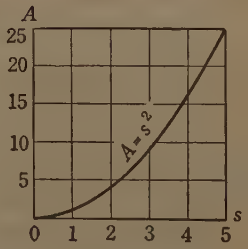
13. If $y = 5x - 3$, supply the missing numbers in the following table:

$x =$	-4	-1	0	3	5			
$y =$						12	37	82

14. Draw the graph of the equation $y = 4x - 1$.

15. The two adjoining graphs picture the formulas $C = 2\pi r$ and $A = s^2$ respectively. From each formula obtain a table and then draw the graphs of the formulas as shown in the diagrams.





16. Draw a graph of the equation $y = x^2 - 4$.

177. Variables and Functions. In a formula such as $d = 4t$, both d and t are *variables*. If we assign values to t, we obtain corresponding values of d. Hence we say that t is the *independent variable*, while d is the *dependent variable*. Similarly, if $y = 5x$, we may regard x as the independent variable, while y is the dependent variable.

Whenever two variables are related in such a way that from every admissible value of one we may obtain *one or more values of the other, the dependent variable is said to be a function of the independent variable.*

Thus, in the case of $d = 4t$, d is a *function* of t; and in the case of $y = 4x + 3$, y is a *function* of x.

It will readily be seen that, in a formula containing two or more variables, any one of these variables may be considered a function of the other variables. In like manner, every algebraic expression is a *function* of all the variables which it contains.

Thus, if $d = 4t$, we also have $t = \frac{d}{4}$; that is, we may regard t as a *function* of d.

2. Ratio and Proportion

178. Numerical Comparisons; Ratio. Suppose that John earns two dollars a week, while Henry earns six dollars a week. We can then compare the earnings of the two boys in several ways. For example, we may say that

(1) Henry earns *four dollars more* than John; or that
(2) Henry earns *three times as much* money as John.

When we compare two numbers by stating *how many times one contains the other*, we are using the *ratio method.*

Thus, the *ratio* (relation) of 6 to 2 is $\frac{6}{2}$, or 3.
Similarly, the *ratio* of 20 to 15 is $\frac{20}{15}$, or $\frac{4}{3}$, or $1\frac{1}{3}$.

In general, the ratio of a to b is the quotient $\frac{a}{b}$. The ratio of a to b is also written in the form $a : b$.

It is evident, then, that *ratios may be looked upon as fractions.* Hence they may be operated upon by the rules that we use in the case of fractions.

For example, the ratio of 2 to 4 and the ratio of 1 to 2 are equal, since $\frac{2}{4} = \frac{1}{2}$.

In stating the ratio of two quantities of the *same kind,* we must express them first in terms of the *same unit.* Thus, the ratio of one yard to two feet is *not* $\frac{1}{2}$, but $\frac{3}{2}$, or $1\frac{1}{2}$.

Exercises. *Using Ratios*

1. In the case of each of the following pairs of numbers give the ratio of the first to the second:

(1) 8, 4.
(2) 9, 3.
(3) 75, 25.
(4) 2, 6.
(5) 7, 2.
(6) 24, 36.
(7) 4, 7.
(8) .8, .04.
(9) $\frac{1}{2}$, 4.
(10) 6, $2\frac{1}{3}$.

2. If t and s are in the ratio of 4 to 5, then $\frac{t}{s} =$ ------.

3. What is the ratio of 1 foot to 24 inches?

4. What is the ratio of 4 quarts to 9 pints?

5. What is the ratio of 18 hours to 2 days?

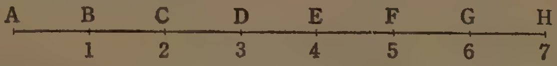
6. Compare the values of the following ratios, indicating the larger one in each case:

(1) $4:6$ and $24:6$.
(2) $3:4$ and $3:2$.
(3) $5:7$ and $1:7$.
(4) $3:4$ and $\frac{3}{2}:4$.
(5) $a:b$ and $\frac{a}{4}:b$.
(6) $a:b$ and $\frac{1}{2}a:2b$.

7. In a crate of 126 oranges 14 are spoiled. What is the ratio of good oranges to the total number in the crate?

8. A recipe for grape jelly calls for 2 pounds of grape juice, $3\frac{1}{2}$ pounds of sugar, and $\frac{1}{2}$ pound of fruit pectin. What is the ratio of sugar to pectin? of grape juice to sugar?

9. In the diagram the points A, B, C, $\cdots$ are separated by equal distances.



Give the numerical value of each of the following ratios if AB represents 1 inch:

(1) $AB:AC$.
(2) $AB:AE$.
(3) $AC:AD$.
(4) $AE:AF$.
(5) $BD:BE$.
(6) $BE:BH$.
(7) $AH:AC$.
(8) $AG:BD$.
(9) $BH:DF$.

10. Express the following ratios in their simplest form:

(1) 18 : 9.
(2) 27 : 6.
(3) 8 : 40.
(4) 75 : 100.
(5) .4 : .8.
(6) .02 : .001.
(7) 1000 : 600.
(8) 2.1 : .42.
(9) $\frac{3}{4} : \frac{1}{4}$.
(10) $2x^2 : 6x^3$.
(11) $a^2 : ab$.
(12) $2^5 : 2^3$.

11. Draw a square having a side of 3 inches, and then divide it into square inches. What is the ratio of two of these small squares to the whole square?

12. On the top of an office desk 30 in. × 40 in. is a blotter 18 in. × 20 in. Find the ratio of the area of the blotter to the area of the top of the desk.

13. From a square piece of sheet iron 14 inches on a side is cut a circular piece having a diameter of 14 inches. Compare the area of the circular piece with the area of the original square piece.

14. The dimensions of a storeroom are 20 feet, 18 feet, and 10 feet. Compare the volume of 100 boxes, each of which is 18 in. × 24 in. × 6 in., with the volume of this room.

15. The Atlantic Ocean covers about 41 million square miles, while the Pacific Ocean has an area of about 69 million square miles. Compare these areas.

16. In each of the following everyday expressions point out the ratio idea which it suggests:

(1) One person out of every four in this town owns an automobile.

(2) Mr. Adams says that he is getting 20 miles from a gallon of gasoline in his new car.

(3) This motor is only 40 % efficient.

(4) In canning fruit, use one cup of sugar for every quart of fruit.

(5) The tax rate this year is \$28 per \$1000 of assessed valuation.

Exercises. *Ratio Problems*

1. Find two numbers whose sum is 100 and whose ratio is 2 : 3.

SUGGESTION. Since $\frac{2}{3} = \frac{2n}{3n}$, we may represent the required numbers by $2n$ and $3n$. Then $2n + 3n = 100$.

2. A line 12 feet long is to be divided into two parts in the ratio of 1 : 3. Find the parts by arithmetic and by algebra.

3. Air is made up principally of oxygen and nitrogen, in the ratio of 1 : 4. In a room containing 10,000 cubic feet of air, how many cubic feet of oxygen are there?

4. Find three numbers whose sum is 1000 and whose ratio is 2 : 3 : 5.

SUGGESTION. Represent the required numbers by $2n$, $3n$, and $5n$.

5. A street is to be paved. Three contractors apportion the work among themselves in the ratio of 2 : 4 : 6. If the length of the street is 1200 feet, for how many feet is each contractor responsible?

6. The enrollment numbers of four courses in a school, called A, C, T, H, were found to be in the ratio 2 : 3 : 4 : 6. Draw a circle graph showing the relative enrollment in each course.

SUGGESTION. Divide 360° into four parts, using the indicated ratio numbers.

7. A firm makes a profit of $10,000 at the end of a year. If all together there are five stockholders, who contributed capital in the ratio of 1 : 3 : 4 : 5 : 7, how much of the profit does each receive?

8. A certain grade of milk was found to contain 4% butter fat, 3.3% protein, 5% sugar, and 1.4% mineral

matter, the rest being water. In 50 pounds of milk there are how many pounds of butter fat? of water?

9. If a team wins 27 games out of 50 games, what is its standing?

SUGGESTION. Express the ratio 27 : 50 in *decimal* form.

10. If a team wins 19 games and loses 21 games, what is its standing? If it loses 8 out of 45 games, what is its standing?

11. A grocer mixed two grades of coffee, using 40 pounds of one and 60 pounds of the other, to make a mixture of 100 pounds. If a man buys 30 pounds of this mixture, how many pounds of each grade does he receive?

12. For the maintenance of a county road, which costs $900 annually, the county pays one third. The remainder has to be paid by three townships according to the population which each has. How much does each have to pay if the townships have 500, 640, and 360 inhabitants respectively?

13. Contractors often use a "1–3–5 mixture" of concrete, which means 1 part cement, 3 parts sand, and 5 parts gravel. How many pounds each of cement, sand, and gravel would be necessary for 2 tons of such a mixture?

14. During a certain year an estate yielded a total net income of $30,000. Of this sum $15,000 came from corporation dividends, $10,000 from nontaxable bonds, and $5000 from all other investments. One of the beneficiaries of the estate received $3000 at the end of that year. In making out his income-tax returns, how much of that sum must he assign to each of the three sources of income named above?

179. Equal Ratios; Proportion. It is evident that any fraction, or ratio, has many equivalent forms.

Thus, $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{5}{10} = \frac{7}{14}$, and so on.

> *A statement expressing the equality of two ratios is called a proportion.*

Thus, $\frac{1}{2} = \frac{7}{14}$ is a proportion. This proportion may also be written in the form $1:2 = 7:14$. It is then read "1 is to 2 as 7 is to 14."

180. The Rule of Proportion. Any convenient proportion may serve to illustrate an important relation existing between the *terms* of a proportion.

(1) $\frac{1}{2} \times \frac{3}{6}.$ Observe that $1 \times 6 = 2 \times 3$.

(2) $\frac{3}{4} \times \frac{9}{12}.$ Observe that $3 \times 12 = 4 \times 9$.

Products such as 3×12 and 4×9, in the second proportion above, are often called *cross products*, as shown by the crossed lines. From these illustrations we may infer the following convenient rule:

In a proportion the cross products are equal.

We can easily show that this rule applies in the case of *any* proportion.

Thus, if $$\frac{a}{b} = \frac{c}{d},$$

we may multiply both members by bd.

Then $$\frac{a}{b} \cdot bd = \frac{c}{d} \cdot bd.$$

Simplifying, we have $$ad = bc.$$

If a proportion is written in the form $a:b = c:d$, the rule may be stated in the following form:

> *In a proportion the product of the outer terms equals the product of the inner terms.*

Illustrative Example 1. Is the following proportion correct: $\frac{4}{5} = \frac{12}{15}$?

Solution. Since $4 \times 15 = 5 \times 12$, the statement is correct.

Illustrative Example 2. If $\frac{7}{8} = \frac{n}{5}$, find n.

Solution. Since $8n = 35$, $n = 4\frac{3}{8}$. (Explain.)

It is obvious that we can also treat a proportion as a *fractional equation.* Hence the value of n, in the above example, could readily be found by clearing the equation of fractions and solving the resulting equation.

Exercises. *Using Proportions*

Test the correctness of each of the following proportions:

1. $\frac{7}{5} = \frac{14}{10}$.
2. $\frac{5}{9} = \frac{25}{50}$.
3. $\frac{10}{17} = \frac{40}{70}$.
4. $\frac{217}{315} = \frac{112}{174}$.
5. $\frac{1}{2} : \frac{1}{4} = 18 : 9$.
6. $\frac{4a}{5b} = \frac{12a}{15b}$.

Find the missing number in each of the following proportions:

7. $\frac{3}{4} = \frac{n}{20}$.
8. $\frac{7}{9} = \frac{n}{10}$.
9. $\frac{n}{3} = \frac{5}{8}$.
10. $\frac{5}{n} = \frac{9}{10}$.
11. $\frac{4.1}{5.2} = \frac{7}{n}$.
12. $\frac{a}{3b} = \frac{x}{4c}$.

Write each of the following equations in the form of a proportion:

13. $ax = bc$.

Solution. We may obtain several proportions, such as

(1) $a : b = c : x$; (2) $a : c = b : x$; (3) $b : x = a : c$.

14. $cy = mp$.
15. $hw = bd$.
16. $lw = LW$.
17. $(a + 2)(c - 1) = ac$.
18. $a^2 - b^2 = 5x$.
19. $x^2 = bh$.

181. Solving Problems by Proportion. Many important problems may be solved quickly by using the *rule of proportion.* In all these problems there must be four numbers which can be so arranged as to form a proportion. To make this clear, we shall consider a simple illustration that will suggest various types of problems that may arise.

Suppose that sugar can be bought at 7 cents a pound. We may then obtain a table such as the following:

Number of pounds . .	1	2	3	4	5	6
Total cost.	7	14	21	28	35	42

Problem 1. If I have to pay 21 cents for 3 pounds of sugar, how much must I pay for 5 pounds?

Solution. This problem can evidently be solved by ordinary arithmetic, by finding out, first, how much *one* pound costs. Instead, we may tabulate the conditions of the problem as follows:

Pounds . .	3	5
Cost . . .	21	

Let $c =$ the required cost, in cents.

Then $3c = 5 \times 21.$

That is, $c = 35.$

Problem 2. If I have to pay 21 cents for 3 pounds, how many pounds can I buy for 35 cents?

Solution. Using a table, we have

Pounds . .	3	
Cost . . .	21	35

Let $p =$ the required number of pounds.

Then $21p = 3 \times 35.$

That is, $p = 5.$

Problem 3. How much will 3 pounds cost if 5 pounds cost 35 cents?

Solution. Using a table, we have

Pounds . .	3	5
Cost . . .		35

In this case, $5c = 3 \times 35.$

That is, $c = 21.$

Problem 4. How many pounds can be bought for 21 cents if 5 pounds cost 35 cents?

Solution. Using a table, we have

Pounds . .		5
Cost . . .	21	35

In this case, $35p = 5 \times 21$.
That is, $p = 3$.

Exercises. *Problems involving Proportion*

1. Suppose that an automobile travels at the rate of 20 miles an hour. If h represents the number of hours it travels, while d represents the corresponding distance, state in words the problem suggested by each of the following tables. Solve each of these problems.

(1)

$h =$	3	5
$d =$	60	

(3)

$h =$	3	
$d =$	60	140

(2)

$h =$		6
$d =$	80	120

(4)

$h =$		5
$d =$	40	100

SUGGESTION. Problem (1) reads: If in 3 hours I can travel a distance of 60 miles, how far can I travel in 5 hours?

2. If 8 bushels of potatoes cost \$7.60, how much would 14 bushels cost?

3. I paid \$5.40 for 12 yards of ribbon. How much would 20 yards cost?

4. How many yards of cloth can be bought for \$29.70 if 9 yards cost \$24.30?

5. If x pounds of coffee cost y dollars, how many dollars would n pounds cost?

6. Two numbers are to each other as 4 is to 5. The first number is 32. What is the second?

7. The larger of two numbers is 21, and the numbers are in the ratio of 7 : 9. What is the smaller number?

8. Mrs. Jones found that she had used 9 pounds of sugar in 5 weeks. At that rate, how many pounds will she use in 6 months?

9. An iron tube 3 feet long was found to weigh 105 pounds. What would be the weight of a tube of the same kind 20 feet long?

10. If an airplane consumed 40 gallons of gasoline in going 370 miles, how much gasoline would it require for a trip of 3200 miles?

11. Mr. Brown has a cylindrical cistern which is 12 feet deep and holds 400 gallons of water. How many gallons of water are in the cistern when the water is 5 feet deep?

12. Two numbers are in the ratio of 2 : 7. When 2 is added to each of the numbers, the ratio of the resulting numbers is 2 : 5. What are the numbers?

13. Two numbers are in the ratio of 8 : 9. When 4 is subtracted from each of the numbers, the ratio of the resulting numbers is 6 : 7. Find the numbers.

14. If the taxes on a house that is assessed for $7000 are $189, what would be the taxes on a house assessed for $8000?

15. A druggist's recipe for making tooth powder calls for 20 pounds of powdered chalk and 3 pounds of pure powdered soap. How much chalk and how much soap does he need to make 46 pounds of tooth powder?

16. A plumber and his apprentice were on a certain job an equal length of time and received \$25.20 for their work. How must this sum be divided between them if the plumber receives \$1.50 an hour and the apprentice 60 cents an hour?

182. Scale Drawing. In designing a house, an architect cannot draw the plans full size. Instead, he makes a scale drawing which shows in *reduced* form the dimensions of the building to be constructed. For example, a distance of one foot may be represented in such a drawing by an inch, or by a convenient fraction of an inch. In like manner, surveyors and draftsmen draw nearly all their plans "to scale."

Suppose that it is necessary to make a scale drawing of a door which is 8 ft. high and 3 ft. wide. If we decide to use a scale of 1 in. to 1 ft., it is evident that we must make a rectangular plan 8 in. long and 3 in. wide. Such a scale is indicated by writing "Scale, 1 in. to 1 ft." or "Scale, 1 in. = 12 in." Other designations are also used, such as "Scale, 1 to 12" or "Scale 1:12" or "Scale $\frac{1}{12}$."

Scale drawing is thus readily seen to be an illustration of the applied use of *ratios*.

Exercises. *Scale Drawing*

1. Draw a line 6 in. long. Then draw it to scale, using in succession the scales 1:2; 1:3; 1:4; 1:8; 1:12.

2. In a scale drawing a certain line was found to be 3 in. long. What was the actual length represented by the line if the scale used was 1:2? 1:3? 1:10? 1:12?

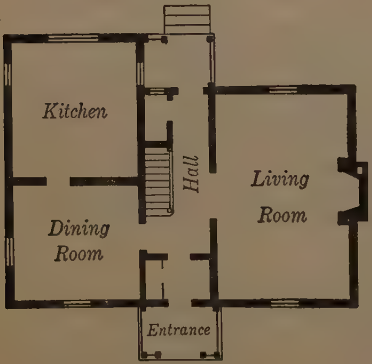
3. A house is to be 24 ft. wide and 32 ft. long. If the scale to be used in the architect's plan is to be $\frac{1}{8}$ in. to 1 ft., what must be the dimensions of the ground plan?

4. A living room was found to be 14 ft. long and 12 ft. wide. What must be the dimensions of a scale drawing representing it, the scale being $\frac{1}{4}$ in. to 1 ft.?

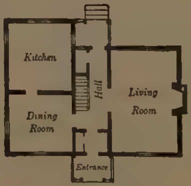
5. Draw any rectangle. Then copy it, using the scale $\frac{1}{2}$. (This means that the dimensions of the *scale drawing* are to be half as large as those of the given rectangle.) Compare the area of the second rectangle with that of the first.

6. Draw a square and then enlarge it, using the scale 3 : 1. How does the area of the second square compare with that of the given square?

183. Similarity. Two figures which differ merely in the scale used in drawing them are said to be *similar*.



(1)



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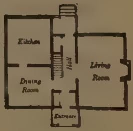
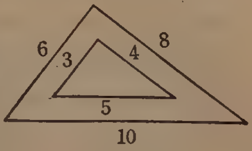


(3)

Thus, (2) and (3) were obtained from (1) by using the scales $\frac{1}{2}$ and $\frac{1}{3}$ respectively. The three figures are *similar.*

Two similar figures have the same *shape*, but not the same *size*. If one figure is an exact copy, or duplicate, of another, the two figures are said to be *congruent*. In that case we may say that the scale used is 1 : 1.

An inspection of the two similar triangles in the diagram will serve to bring out the important fact that their corresponding *angles* are equal, but not their corresponding *sides*.



The sides of two similar triangles are said to be *proportional*. This means that *the ratio of any two corresponding sides of two similar triangles equals the ratio of any other pair of corresponding sides.*

For example, the triangles shown above furnish the following set of equal ratios: $\frac{3}{6} = \frac{4}{8} = \frac{5}{10}$.

We may also say, however, that *the ratio of any two sides of the first triangle equals the ratio of the corresponding sides of the second triangle.*

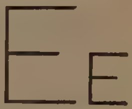
Thus, in the case of the triangles above, we may obtain proportions such as the following: $\frac{3}{4} = \frac{6}{8}$, $\frac{3}{5} = \frac{6}{10}$, and so on.

In general, *similar figures may be defined as figures that have their corresponding angles equal and their corresponding sides proportional.*

The study of similarity is of great importance. Every figure drawn to scale is similar to the figure which it represents. The use of blue prints is found indispensable by builders, machinists, engineers, and designers. Without maps and charts surveyors, automobilists, navigators, and air pilots would face a serious handicap. A moving picture is a series of enlargements of the views appearing in the film.

Similarity is equally important in the case of solids. In many industries it is necessary to manufacture several "sizes" of an article, based on the same pattern or model. For example, in the case of shoes, hats, wearing apparel, kitchen utensils, hardware, furniture and the like, dealers must have at hand all the usual sizes which are likely to be in demand. When we look at a small object through a microscope, we see an enlarged view which is really similar in outline to the original object. In a printing office it is customary to have the same style of type in various sizes. Many other applications of similarity will readily suggest themselves.

Exercises. *Similar Figures*



1. Make a drawing representing the capital letter *E*. Then draw three similar letters, using in succession the scales 1:2, 3:1, and 1:4.

2. Show that any two squares are similar.

3. Two figures, F_1 and F_2, are similar. The table gives the sides of F_1. Find the corresponding sides of F_2 if the side corresponding to 8 is 2.

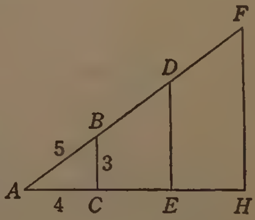
F_1	8	9	10	12	15	20
F_2	2					

4. The dimensions of a rectangle are 5 and 17. In a similar rectangle the side corresponding to 5 is 12. Find the side corresponding to 17.

5. If the three triangles in the adjoining diagram are similar, show that

$$(1)\ \frac{BC}{AC}=\frac{DE}{AE};\quad (2)\ \frac{BC}{DE}=\frac{AC}{AE}.$$

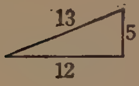
Read other proportions suggested by the diagram.

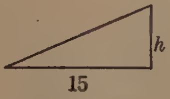


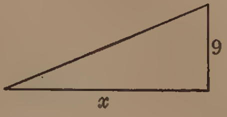
6. The sides of triangle *ABC*, in the diagram, are 3, 4, and 5. (1) If $AE = 10$, find *DE*. (2) If *FH* is 15, find *AH*.

7. To find the height of a tree, Fred Brown first measured its shadow, which he found to be 80 feet long. He then set up a pole in a vertical position. Its height above the ground was 4 feet and its shadow at that hour was found to be 5 feet. From these measurements Fred determined the height of the tree. Can you explain his method? What was the height of the tree?

8. If the three triangles shown below are similar, find the value of h and of x.







3. VARIATION

184. Proportional Sets of Numbers; Direct Variation. The following table is based on the formula $p = 4s$:

$s =$	1	2	5	7	9	10	15
$p =$	4	8	20	28	36	40	60

From this table we may readily obtain an indefinite number of proportions. The equal ratios may be selected either *vertically* or *horizontally*.

Thus, $\frac{1}{4} = \frac{2}{8}$, $\frac{5}{20} = \frac{7}{28}$, $\frac{2}{8} = \frac{10}{40}$, and so on.
Also, $\frac{1}{2} = \frac{4}{8}$, $\frac{2}{5} = \frac{8}{20}$, and so on.

That is, *the ratio of any pair of corresponding values of* s *and* p *is equal to the ratio of any other pair of corresponding values.*

Again, *any two values of* s *have the same ratio as the corresponding values of* p.

Hence we say that the value of p is *directly proportional* to the value of s, or that p *varies directly as* s. For example, if we multiply a value of s by 5, we also multiply the corresponding value of p by 5.

The relations explained above may be represented in more general form as follows:

$s =$	s_1	s_2	s_3	s_4	s_5	$\cdots$
$p =$	p_1	p_2	p_3	p_4	p_5	$\cdots$

(Observe that the subscripts in this table serve to indicate corresponding values of s and p. Thus, s_1 corresponds to p_1, s_2 corresponds to p_2, and so on.)

If the corresponding values of s and p are so related that

$$\frac{s_1}{p_1} = \frac{s_2}{p_2} = \frac{s_3}{p_3}, \text{ and so on,}$$

or that

$$\frac{s_1}{s_2} = \frac{p_1}{p_2}, \; \frac{s_2}{s_3} = \frac{p_2}{p_3}, \text{ and so on,}$$

then p varies directly as s. Evidently, we may also say that s varies as p.

In general, *if any two variables, such as y and x, are so related that the ratio $\frac{y}{x}$ is always the same, or remains constant, then y is directly proportional to x, or y varies directly as x.* In symbols, if $\frac{y}{x} = c$, or if $y = cx$, where c has a *constant* value, then y varies directly as x.

It follows, then, that any formula of the type $y = cx$ represents a case of *direct variation*. This is true, for example, of the formulas $c = \pi d$, $d = 10\,t$, $c = 10\,p$, $c = .05\,p$, and so on.

Whenever we are concerned with a *constant rate*, or a *uniform change*, we have a case of *direct variation* of the type explained above. Many of the arithmetical problems of everyday life, as well as numerous situations arising in science, industry, and engineering, serve to illustrate the common occurrence of direct variation.

There are, however, other important types of variation, some of which will be suggested in the exercises that follow.

Exercises. *Variation*

1. Show that the ratio of y to x in the following table is constant. Write the formula which expresses the relation of y to x.

$x =$	2	5	7	8	9	10
$y =$	16	40	56	64	72	80

2. In each of the following tables one variable is directly proportional to the other. Supply the missing numbers in each table.

(1)

$p =$	3	6		9		60
$q =$	5		10		20	

(2)

$r =$	10	20		60		100
$s =$	11		33		77	

(3)

$t =$	5	7	8		11	
$v =$		91		130		169

3. If gasoline sells at 23¢ a gallon, make a table giving the cost of any number of gallons from 1 to 10. Show that the cost (c) is directly proportional to the number of gallons (g) bought. Write the formula which shows the relation of c to g.

4. Using the method of Exercise 3, show that the distance (d) covered by an automobile which travels at the rate of 25 miles per hour is directly proportional to the number of hours (h) it travels.

5. If a clerk earns \$40 a week, show that his income (i) varies directly as the number of weeks (w) he works. If $w = 10$, find i; if $i = \$280$, find w.

6. Suppose that a printing press turns out 200 complete newspapers per minute. Show that the total number of papers (p) is proportional to the number of minutes (m) the press is in operation. If $m = 45$, find p.

7. Give other illustrations of direct variation based on the idea of (1) simple interest; (2) wages; (3) insurance; (4) discount; (5) profit or loss; (6) commission; (7) taxes.

8. Suppose that y varies directly as x. If two corresponding values of y and x are 45 and 3 respectively, what is the ratio of y to x? In that case find y when x is 7.

EXPLANATION. From our study of linear equations (Chapter X) it is evident that any formula of the type $y = cx$, where c is a constant number, has for its graph a *straight line*. Moreover, it is obvious that this straight line passes through the *origin*. (Why?) Hence any formula which illustrates direct variation of the type explained above can be pictured by a *straight line*.

9. Show that the graph of each of the following formulas is a straight line: (1) $d = 5\,t$; (2) $c = 10\,p$; (3) $i = 7\,d$; (4) $i = .05\,p$; (5) $C = \pi d$; (6) $p = 6\,s$.

10. Show that the circumferences of two circles have the same ratio as their radii.

SUGGESTION. $\frac{C_1}{C_2} = \frac{2\,\pi r_1}{2\,\pi r_2} = \frac{r_1}{r_2}.$

11. The radii of two circles are 7 inches and 21 inches respectively. Compare their circumferences.

12. Two circular running tracks have diameters of 300 feet and 600 feet respectively. How do they compare in length?

13. The diameter of the moon is approximately one fourth as large as that of the earth. How does the moon's equator compare with that of the earth?

14. The diameter of the sun is about 110 times as large as that of the earth. The earth's equator is about 25,000 miles. What is the approximate length of the sun's equator?

15. In the formula $p = 6\,w$, explain how p changes when w is replaced by (1) $2\,w$; (2) $5\,w$; (3) $\frac{w}{2}$; (4) $.1\,w$.

16. If $C = \pi d$, state how C changes when d is replaced by (1) $3\,d$; (2) $10\,d$; (3) $\frac{d}{2}$; (4) nd.

17. In the equation $y = 4x - 3$, does y become larger or smaller as x increases?

18. In the formula $i = prt$, how does i change when p is replaced by $2p$? when r becomes $3r$? when t becomes $\frac{t}{2}$? when p becomes $3p$ and r becomes $2r$?

19. In the formula $A = lw$, how does A change when l is replaced by $4l$? when w becomes $5w$? when l becomes $3l$ and w becomes $4w$?

EXPLANATION.* Two variables, such as y and x, may be so related that y varies directly as the *square* of x. This means that the ratio $\frac{y}{x^2}$ is a constant number c, or that $y = cx^2$. In such a case we can easily show that $\frac{y_1}{y_2} = \frac{x_1^2}{x_2^2}$, $\frac{y_2}{y_3} = \frac{x_2^2}{x_3^2}$, and so on.

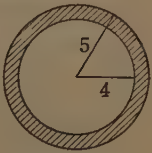
20. Show that the area of a square varies directly as the *square* of its side.

SUGGESTION. Using the formula $A = s^2$, make a table of corresponding values of s and A.

21. Show that the areas of any two circles have the same ratio as the *squares* of their radii.

22. The radii of two circles are 2 in. and 3 in. respectively. Show that the area of the smaller circle is $\frac{4}{9}$ that of the larger circle.

23. The radii of the two circles in the figure are 4 and 5 respectively. Compare the area which is shaded with that of the larger circle.



24. In the formula $S = 6e^2$, explain how S changes when e is replaced by (1) $2e$; (2) $5e$; (3) $\frac{e}{2}$; (4) ne.

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* The exercises which follow may be postponed or omitted, at the discretion of the teacher.

25. Show that the volume of a cube is directly proportional to the *cube* of its edge.

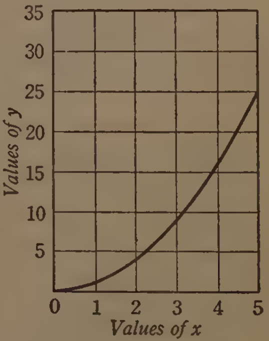
26. The formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$. How do the volumes of two spheres compare in size?

27. The diameters of two spheres are 4 and 6 respectively. Compare their volumes.

28. In the formula $V = e^3$, explain how V changes when e is replaced by (1) $2e$; (2) $3e$; (3) $\frac{e}{2}$; (4) $10e$.

29. Two cylindrical tanks have the same height, but the radius of one is twice as large as that of the other. Compare (1) the areas of their bases; (2) their volumes.

30. Whenever we have a formula of the type $y = cx^2$, its graph is a curved line such as the one shown in the adjoining diagram. Verify this statement by drawing the graph of one of the following formulas: (1) $A=2s^2$; (2) $A=\pi r^2$; (3) $s=16t^2$; (4) $T = 15s^2$; (5) $S = 6e^2$.



y varies as x^2

185. Inverse Variation.* The time it takes to cover a given distance evidently depends on the rate at which one travels. The faster one travels, the *less* time is required. Thus, if the distance is 60 miles, the relation between the rate (r), in miles per hour, and the time (t), in hours, may be shown by a table such as the following:

$r =$	3	4	5	6	10	15	30
$t =$	20	15	12	10	6	4	2

---

* This section may be postponed or regarded as optional.

Observe that if r is *multiplied* by any number, say 5, t is *divided* by that number. Also, the product of any two corresponding values of r and t is constant, namely 60.

Moreover, *the ratio of any two values of r is equal to the inverted ratio of the corresponding values of t.*

Thus, $\frac{3}{4} = \frac{15}{20}$, $\frac{4}{5} = \frac{12}{15}$, and so on.

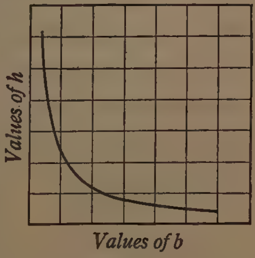
Hence we say that t *varies inversely as r*, or that t *is inversely proportional to r*. In general, *y varies inversely as x, or is inversely proportional to x, if the product xy is constant.*

In symbols, y varies inversely as x if $xy = c$, or if $y = \frac{c}{x}$, or if $\frac{y_1}{y_2} = \frac{x_2}{x_1}$, $\frac{y_2}{y_3} = \frac{x_3}{x_2}$, and so on.

Exercises. *Inverse Variation*

1. Express in symbols the fact that the number of pounds of coffee that can be bought for $6 varies inversely as the price per pound.

2. Show that the number of days necessary to complete a certain job varies inversely as the number of men employed to do the work.



3. Robert Hall wished to make a rectangular garden plot having an area of 60 square feet. Make a table suggesting various possible dimensions he might use. Show that the height (h) is inversely proportional to the base (b). Draw a graph picturing this relation, as suggested in the adjoining diagram.

4. Suppose that y varies inversely as x, and that when $x = 20$, $y = 5$. Find y when $x = 50$.

5. If gas is put under pressure, it can be shown that the volume of the gas varies inversely as the pressure, so long as the temperature is constant. Suppose that 6 cubic feet of gas in a cylinder are under a pressure of 20 pounds per square inch. If the pressure is increased to 40 pounds per square inch, what will be the corresponding volume of the gas in the cylinder?

6. The *law of the lever*, as explained on page 354, may be stated in the form $W_1d_1 = W_2d_2$. Show that this law represents a case of inverse variation.

EXPLANATION. When $y = \frac{c}{x^2}$, we say that y varies inversely as the *square* of x.

7. If $y = \frac{36}{x^2}$, explain how y changes when x is replaced by (1) $2x$; (2) $3x$; (3) $\frac{x}{2}$; (4) $.1x$.

8. The intensity (brightness) of a light varies inversely as the *square* of the distance from the source of light. A reader first holds a book 3 feet away from a lamp and then 6 feet away. How has the intensity of the light on the printed page been changed?

9. The weight of an object above the surface of the earth varies inversely as the square of the distance from the center of the earth. Express this relation in symbols.

186. Summary. In this chapter you have learned

(1) The importance of related changes in everyday life.

(2) How to express numerical relationships and cases of dependence.

(3) The meaning of *variable, constant, independent variable, dependent variable, function.*

(4) How to use ratios in numerical comparisons.

(5) How to use proportions in the study of similarity and in other applied problems.

(6) The meaning and use of various types of variation.

Test on Chapter XIV

1. Complete correctly each of the following statements:

(1) A ratio is a numerical comparison of two quantities by means of the process of -----.

(2) An equality of two ratios is called a -----.

2. Write three other ratios each of which has the same value as 4 : 6.

3. Test the correctness of each of the following proportions:

(1) $1.2 : 3 = 8 : 15.$ (2) $\frac{3a}{4b} = \frac{1.5a^2}{2ab}.$

4. Find the missing number (x) in each of the following proportions:

(1) $\frac{5}{7} = \frac{6}{x}.$ (2) $6x : 4c = 3r : 5s.$

5. A rectangle 10 feet long and 4 feet wide is similar to a second rectangle whose length is 25 feet.

(1) Find the width of the second rectangle.

(2) What is the ratio of the perimeter of the smaller rectangle to that of the larger?

(3) What is the ratio of the area of the larger rectangle to that of the smaller?

6. Two partners in business invested capital in the ratio of 3 : 5. The net profit for one year was $24,000. What was each man's share of the profit, if it was divided in the same ratio as the original capital?

7. The annual premium on a \$2500 life insurance policy was \$70. At the same rate, what would be the premium on a policy for \$6000? (Solve by proportion.)

8. The cost of electricity per month, in a home, depends upon the amount of electricity used.

(1) Supposing the cost of electricity to be 8 cents per kilowatt hour, write a formula which shows the relation of the cost (c) to the amount of electricity used (e), if e is expressed in kilowatt hours.

(2) Make a table which shows the relation of c to e for the following values of e: 20, 25, 36, 40, 50.

9. By referring to the table in Exercise 8, show that the cost is *directly proportional* to the amount of electricity used.

10. In the formula $V = lwh$, how does V change (1) when l is replaced by $3l$? (2) when h becomes $4h$? (3) when w becomes $2w$ and h becomes $3h$?

CHAPTER XV

NUMERICAL TRIGONOMETRY

1. THE DIRECT MEASUREMENT OF LINES AND ANGLES *

187. Importance of Measurement. It has been said that measurement is the *master art.* All the practical arts, as well as industry, commerce, and science, depend on accurate measurement. In fact, modern civilization would be impossible without the art of precise measurement.

188. Two Kinds of Measurement. Many measurements can be made *directly.* Thus, in finding the area of a rectangular floor, you measure *directly* the length and the width of the floor. There are many measurement problems, however, which can be solved only *indirectly.* Thus, when a surveyor wishes to determine the distance between two points on the ground, there may be obstacles, such as trees, buildings, or bodies of water, that prevent direct measurement. Such a situation is suggested by the diagram which is shown at the top of the following page.

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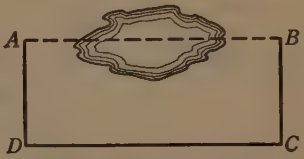
* Pupils who have completed a course in intuitive geometry usually have no difficulty in mastering simple trigonometric problems. Such pupils may omit pages 397–406. For those who lack this foundation some introductory geometric work should be provided. This work should include (1) a careful discussion of the necessary geometric terms; (2) exercises in the measurement of lines and angles; (3) the informal study of the essential properties of triangles. Teachers who wish to supplement the preliminary exercises given in this chapter may be referred to the following sources:

Geometry for Junior High Schools, by William Betz. Published by J. M. Egloff, Rochester, New York.

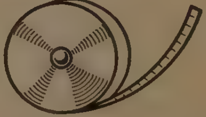
Work Book in Intuitive Geometry, by Betz, Miller, and Miller. Published by The Harter School Supply Company, Cleveland, Ohio.

The distance AB cannot be measured directly. If, however, a surveyor constructs a rectangle $ABCD$ as shown here, he can measure CD and thus find the length of AB *indirectly*.



In this chapter we shall study simple problems involving the *indirect* measurement of heights and distances. For this purpose we shall need to review first some of the processes of *direct* measurement of lines and angles, as well as certain properties of triangles, with which you are probably familiar from your previous work in mathematics.

189. Measuring Lengths Directly. Among the most common instruments for the measurement of lengths are the ruler, the yardstick, the meter stick, and the surveyor's tape. Each of these instruments is divided, or "graduated," into smaller divisions. Thus, the ruler shows inches and fractions of an inch. A surveyor's tape is marked in feet and inches or in feet and tenths of a foot. The following exercises will serve to give you some practice in the direct measurement of lines:



Exercises. *Measuring Lengths Directly* *

1. Draw four lines, making each about 5 inches long. On the first mark off a distance of $3\frac{1}{2}$ inches; on the second, of $2\frac{3}{4}$ inches; on the third, of $4\frac{3}{8}$ inches; on the fourth, of $1\frac{7}{16}$ inches.

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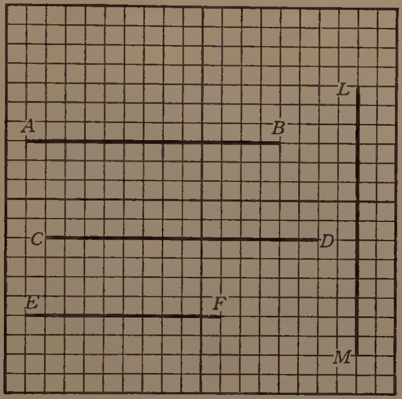
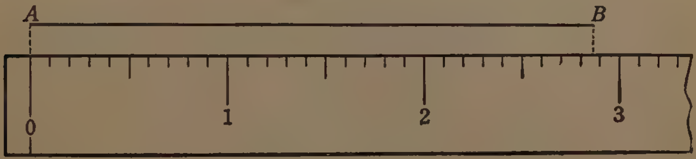
* For exercises of this type, each pupil should be provided with a ruler graduated to sixteenths of an inch, and, if possible, to tenths. Such scales may easily be improvised by the pupils. Squared paper graduated to tenths of an inch will also be found very helpful. Metric scales showing centimeters and millimeters are often more accessible, and may be substituted in a number of

2. The diagram below shows squared paper in which each inch is divided into 10 equal parts. What is the length of *AB*? of *CD*? of *EF*? of *LM*?



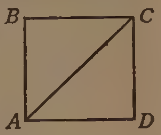
3. Observe that the length of *AB*, in the following diagram, is between 2 and 3 inches. What is the length of *AB* to the nearest inch? to the nearest tenth of an inch?



4. On a strip of paper mark off a scale of 9 inches, dividing each inch into tenths. (To obtain tenths of an inch, you may use the scale shown in Exercise 3.) Measure the length of a page of this book (1) to the nearest inch; (2) to the nearest tenth of an inch.

the exercises. A detailed discussion of the technique of direct measurement, of *approximations* and *degrees of accuracy*, may be found in the author's monograph on "The Teaching of Direct Measurement in the Junior High School," in the Third Yearbook of the National Council of Teachers of Mathematics, Teachers College Bureau of Publications, New York City.

5. On squared paper showing inches, draw a square $ABCD$, making its side AB 3 inches long. Measure the diagonal AC to the nearest tenth of an inch. (You will find that the length of AC is 4.2 inches, very nearly.)

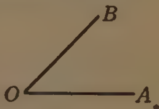


6. In Exercise 5, what is the length of AC if AB is 4 inches long? 5 inches? 6 inches? (The required lengths, in inches, are 5.7, 7.1, and 8.5 respectively.)

7. On squared paper showing inches, plot the following points: $A(2, 1)$; $B(1, 4)$; $C(3, 6)$; $D(5, 6)$; $E(7, 4)$; $F(5, 1)$. Connect these points in succession. Find by measurement the length of each side of the resulting figure, to the nearest tenth of an inch, and thus determine the *perimeter* of the figure.

190. The Angle. When two lines are drawn from the same point, the figure formed is called an *angle*. The point from which the lines are drawn is the *vertex* of the angle, while the two lines are called the *sides*.

In the figure, O is the vertex of the angle, while OA and OB are the sides.



There are three ways of naming or reading angles: (1) A capital letter may be placed near the vertex, as in Fig. 1.

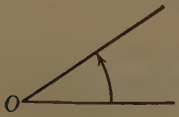


FIG. 1

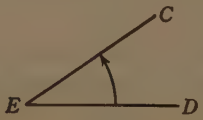


FIG. 2

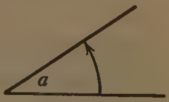
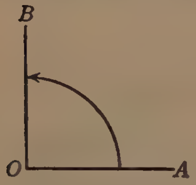


FIG. 3

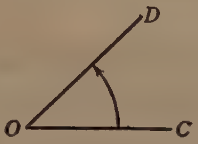
(2) We may also use three letters, placed as in Fig. 2. (3) A small letter may be placed within the angle, as in Fig. 3.

The word "angle" is often abbreviated by using the symbol ∠. Hence the three angles shown above may be referred to as ∠ O, ∠ DEC, and ∠ a respectively. The sides of an angle are sometimes said to *include* that angle.

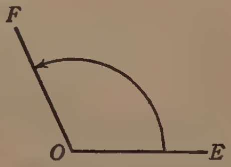
191. Kinds of Angles. In opening a book, you are forming angles of various sizes. Some of these angles are so important that they are given special names. The *right angle* is familiar to you from its constant appearance in such figures as rectangles and squares. An angle which is *less* than a right angle is called an *acute angle.* An angle which is *greater* than a right angle, but less than the sum of two right angles, is called an *obtuse angle.* When one line forms a right angle with another, each line is said to be *perpendicular* to the other. Thus, in the first figure above, OB and OA are perpendicular to each other.



Right angle

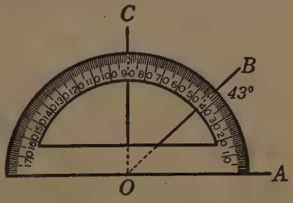


Acute angle



Obtuse angle

192. Measurement of Angles. In measuring the size of an angle, we use a special instrument, the *protractor*, which was described in Chapter VI. With the aid of the equal arc-degrees on the rim of the protractor, we may readily mark off in succession 360 equal angles about any point as a center. Each of these small angles is called an *angle-degree*, or simply a *degree.* The diagram shows how we may obtain an angle of a given number of degrees by using a protractor. Thus, ∠ AOB contains 43°. A *right angle*, such as ∠ AOC, contains 90°.



For very accurate measurements, the degree is divided into smaller divisions called *minutes* and *seconds*. There are 60 *minutes* (′) in one degree, and 60 *seconds* (″) in one minute.

If the sum of two angles is 90°, each angle is said to be the *complement* of the other; if the sum is 180°, each angle is said to be the *supplement* of the other.

Exercises. *Measuring Angles Directly*

1. Placing a protractor on a sheet of paper, mark off in succession angles of 10°, 20°, 30°, and so on, up to 180°.

2. Draw an angle of 43°; of 75°; of 60°; of 89°.

3. In how many minutes does the minute hand of a clock revolve through an angle of 90°? of 30°? of 150°? In how many hours does the hour hand of a clock describe these angles?

4. Draw $\angle a = 80°$; $\angle b = 40°$; $\angle c = 15°$; $\angle d = 90°$.

5. Draw an angle of 40° and then draw its *complement* and also its *supplement*.

6. What is the complement of 30°? of 60°? of 45°? of 80°?

7. What is the supplement of 80°? of 100°? of 10°? of 90°?

8. What is the complement of an angle of $x°$? of $y°$? of $(a + b)°$?

193. Approximate Character of Measurement. All measurements which are concerned with lines or angles are only approximately correct. That is, a small error is usually made, whatever scale is used. As a rule, a person who is skillful in measuring does not make an error larger than half of the smallest division used in the measurement. Thus, if the smallest division used in measuring a line is

$\frac{1}{8}$ inch, the error should not be larger than $\frac{1}{16}$ inch. In all measurements it is desirable to have some idea as to the degree of accuracy which is possible or necessary.

Thus, a carpenter usually makes measurements which are accurate to the nearest sixteenth or thirty-second of an inch. In machine work, however, errors in measurements ordinarily do not exceed a thousandth or a ten-thousandth of an inch. In still more accurate work, precise measurements are now possible that involve errors expressed in millionths of an inch.

194. Rounding off Numbers. Suppose that in a machine shop the length of a metal plate is measured by three apprentices. The first reports it to be 8.2 inches. The second, measuring more accurately, finds it to be 8.24 inches. By using a special measuring instrument, the third finds it to be 8.247 inches. The first measurement represents "two-figure" accuracy; the second, "three-figure accuracy"; and the third, "four-figure accuracy."

It is also customary to speak of such measurements as containing two, three, or more *significant figures*. Thus, 8.2 inches contains two significant figures.

In ordinary measurements it is sufficient to have two-figure or three-figure accuracy. Hence, in any computation, it is usually advisable to "round off" the numbers involved in such a way that we obtain only the desired degree of accuracy. Such "rounded numbers" are always "approximate numbers."

In rounding off a number, the following rules are commonly used:

1. *If the figure which is dropped is greater than 5, the preceding figure should be increased by 1.*

Thus, the value of 7.246, to the nearest 0.01, is 7.25.

2. *If the figure which is dropped is less than 5, the preceding figure remains unchanged.*

Thus, the value of 8.432, to the nearest 0.01, is 8.43.

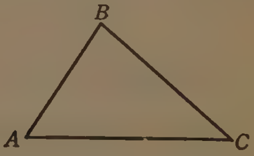
3. *If the figure which is dropped is 5, the preceding figure remains unchanged when it is even, but is increased by 1 when it is odd.*

Thus, 6.385 becomes 6.38, to the nearest 0.01, while 2.735 becomes 2.74, to the nearest 0.01.

195. The Triangle. The triangle is a very important figure. It is used extensively in the construction of houses, bridges, trusses, and the like. In this chapter, however, we shall study the triangle for purposes of *indirect measurement.*

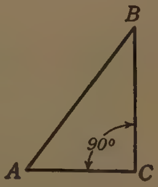
Exercises. *The Triangle*

1. Draw a large triangle. Denote its angles by A, B, and C. Measure each angle with a protractor. Find the sum of the three angles.



2. Repeat Exercise 1, by measuring the angles of three other triangles. Your results will suggest the important truth that *the sum of the angles of any triangle is 180°.*

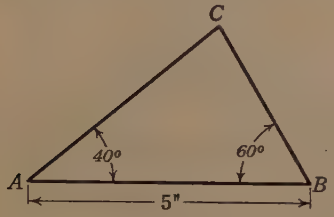
3. A triangle, one of whose angles is a *right* angle, is called a *right triangle.* Draw a right triangle, such as triangle ABC in the diagram. Measure its acute angles, $\angle A$ and $\angle B$. What do you observe?

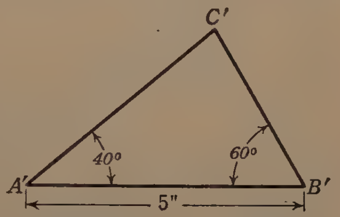


4. If two angles of a triangle are 70° and 50°, how many degrees are there in the third angle?

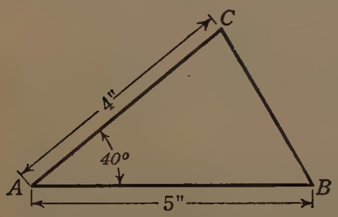
5. Show that every triangle must have two acute angles.

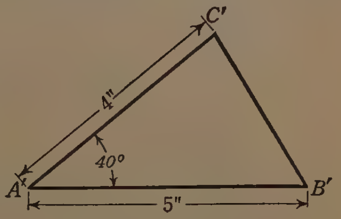
6. Draw a triangle ABC, making $AB = 5$ inches, $\angle A = 40°$, and $\angle B = 60°$. Then draw another triangle, $A'B'C'$, using the same measurements. (A' is read "A-prime," B' is read "B-prime," and so on.) Cut out the two triangles. If your work was done accurately, you will find that one triangle fits exactly on the other. Such triangles are said to be *congruent*. That is, they have the same size and the same shape.





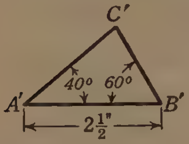
7. Draw a triangle ABC, making $AB = 5$ inches, $\angle A = 40°$, and $AC = 4$ inches. Then draw a second triangle $A'B'C'$, using the same measurements. Cut out the two triangles. As in the preceding exercise, you will find that one fits exactly on the other; that is, the two triangles are *congruent*.

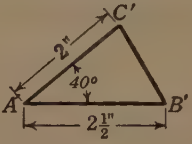




NOTE. In Exercise 6 we copied the triangle ABC by using *three* measurements, namely, one *side* and the two adjoining *angles*. This method is often referred to as the method of *a. s. a.* Similarly, the method used in Exercise 7 is often called the method of *s. a. s.* (Explain.)

8. If, in Exercises 6 and 7, a different scale is used in drawing the second triangle, as shown in the diagrams below, you will find that the second triangle is *similar* to the first; that is, the two triangles have the same *shape* but not the same *size.*





Complete the following statements:

(1) Two triangles are similar if their corresponding angles are ___________.

(2) Two triangles are similar if they have an angle of one equal to an angle of the other and the including sides ________.

196. Important Facts about Triangles. The conclusions suggested by the exercises above may be stated as follows:

1. *The sum of the angles of any triangle is 180°.*
2. *The sum of the two acute angles of a right triangle is 90°. Hence each acute angle is the complement of the other.*
3. *A triangle may be copied by using three measurements, as in the case of a. s. a. or s. a. s.*
4. *If two triangles have their corresponding angles equal, they are similar.*
5. *If two triangles have an angle of one equal to an angle of the other and the including sides proportional, they are similar.*
6. *If two right triangles have an acute angle of one equal to an acute angle of the other, they are similar.*

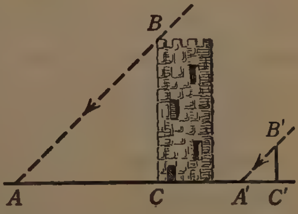
2. INDIRECT MEASUREMENT; TRIGONOMETRIC RATIOS

197. Meaning of Trigonometry. Before people had accurate instruments for measuring angles, they often used congruent or similar triangles in a crude but interesting way to find unknown heights and distances which they could not measure directly. These methods of *indirect* measurement led by slow degrees to a branch of mathematics which is now called *trigonometry*. The word "trigonometry" means "triangle measure."

Trigonometry is used constantly for many practical purposes. In particular, it is the foundation of surveying, navigation, and astronomy.

In the following pages we shall study some of the methods by which numerical trigonometry uses certain known parts of triangles to determine *indirectly* distances or angles that are *not* known.

Preliminary Exercises. *Finding Heights and Distances*



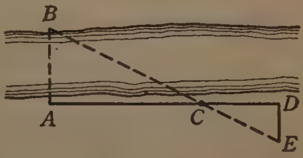
1. In olden times unknown heights were often determined by a process of "shadow reckoning." Suppose that it was desired to find the height of a tower, such as BC in the diagram. A rod $B'C'$ was placed vertically on the ground. Let AC and $A'C'$ represent the shadows of the tower and the rod respectively. Show that when $A'C' = B'C'$, then $AC = BC$.

2. What is the height of a tree whose shadow is 50 feet in length while a rod 3 feet long casts a shadow 5 feet in length?

3. In Exercise 1, what is the height of the tower if $AC = 100$ feet while $B'C' = 3$ feet and $A'C' = 4$ feet?

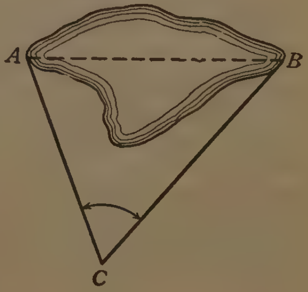
SUGGESTION. The two acute angles at A and A' are practically equal. (Why?) Hence the two triangles ABC and $A'B'C'$ are similar. (Why?) That is, $\frac{BC}{AC} = \frac{B'C'}{A'C'}$.

4. The following method of finding the width of a stream was used by surveyors many centuries ago:



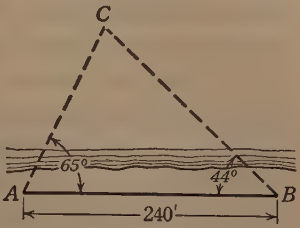
Suppose that A and B are points on opposite sides of a stream. Lay off a distance AC along the bank of the stream at right angles to AB. At C set up a vertical rod. Extend AC to D, making CD any convenient fraction of AC, such as one half. Then walk along the line DE, which makes a right angle with AD, until you find yourself in the line BC extended. Then, by measuring DE, it is possible to find by computation the length of AB; for the triangles ABC and CDE are similar. (Why?) Thus, if $CD = \frac{1}{2} AC$, show that $DE = \frac{1}{2} AB$.

5. In Exercise 4, if $AC = 200$ feet, $CD = 100$ feet, while $DE = 150$ feet, what is the width of the stream?

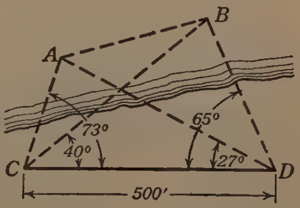


6. In the diagram, AB represents the distance across a pond. To find the length of AB, the following method may be used. From a convenient point C measure the length of CA and CB. Placing a protractor in a horizontal position, determine the size of $\angle C$. It is then possible to draw the triangle ABC to scale and thus to determine AB. If $CA = 200$ feet, $CB = 250$ feet, and $\angle C = 60°$, find AB.

7. A ship was anchored at a distance from the shore. A Boy Scout stationed at A found the approximate distance from A to C, the position of the ship, by *three* measurements. Can you suggest his method? Using the measurements shown in the figure, find by means of a scale drawing the approximate length of AC.



8. Two ships, A and B, were anchored at some distance from the shore. A group of Boy Scouts wished to find the distance between these ships. They made the *five* measurements that are shown in the diagram, CD being a line on the shore. By using these measurements the boys determined the approximate distance from A to B. Can you explain their method? How long is AB?



198. Measuring Angles Out of Doors. For measuring lines surveyors use a *tape*, and for measuring angles they use a *level* or a *transit.*

If the sides of an angle are in a horizontal plane, it is called a *horizontal angle*; if the sides are in a vertical plane, the angle is called a *vertical angle.*

A *transit* * consists essentially of two graduated circles (or circular protractors) and a telescope which is so adjusted that it can revolve in either a horizontal or a vertical plane.

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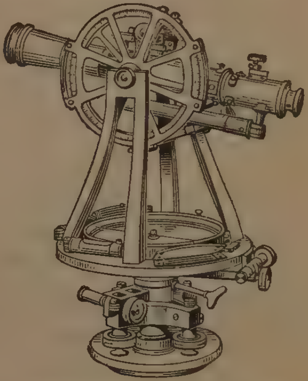
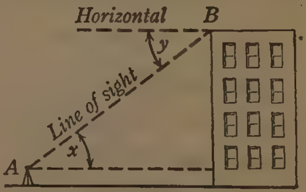
* A substitute for a transit may readily be made by the pupils by using a short length of pipe and ordinary protractors. Valuable suggestions concerning the use of surveying instruments may be found in a monograph on "The Use of Measuring Instruments in Teaching Mathematics," in the Third Yearbook of the National Council of Teachers of Mathematics, by C. N. Shuster.

By turning the telescope up and down it is possible to measure *vertical angles* by reading the scale on the *vertical* circle. Similarly, a *horizontal angle* may be measured by using the scale on the *horizontal* circle.

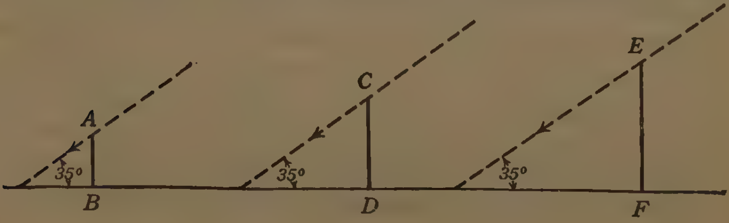


The figure below will illustrate two common measurements that arise in the case of *vertical* angles. A person standing at A must look *upward* if he wishes to see the point B, while a person standing at B must look *downward* in order to see the point A.

In order to measure $\angle x$ with a transit, the telescope must be turned upward, from a horizontal position, until the line of sight passes through B. Hence $\angle x$ is called an *angle of elevation*. Similarly, $\angle y$ is said to be an *angle of depression*. It can easily be shown that $\angle x = \angle y$. In practice, allowance must be made for the height of the transit in the measurement of vertical angles.

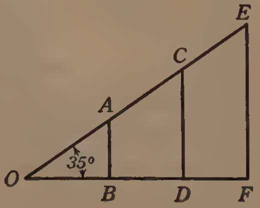


199. An Important Ratio; the Tangent. Suppose that AB, CD, and EF are vertical rods. In the diagram their



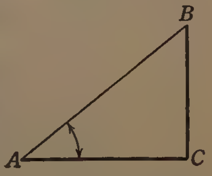
shadows are shown at the time when the rays of the sun make an angle of 35° with the ground. The three triangles outlined above are evidently similar, since they are right triangles with an acute angle the same in all. It would therefore be possible to place these triangles in the positions suggested by the adjoining figure. Since the sides of these triangles are proportional, we have

$$\frac{AB}{OB}=\frac{CD}{OD}=\frac{EF}{OF}.$$



Thus, if $\angle O=35°$ and $OB=10$, you will find that $AB=7$, very nearly. Hence the ratio of the corresponding sides of the other right triangles containing $\angle O$ will also be $\frac{7}{10}$, or 0.7.

It follows, therefore, that in a right triangle such as ABC in the figure at the right, *the ratio* $\frac{BC}{AC}$ *does not depend on the size of the triangle, but merely on the size of* $\angle A$. So long as the value of $\angle A$ is unchanged, the ratio $\frac{BC}{AC}$ is always the same. This constant ratio is called the *tangent of* $\angle A$, which is usually abbreviated by writing "tan A."

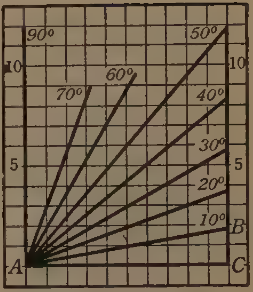


In the right triangle ABC shown above,

AB is called the *hypotenuse* of the triangle,
BC is the *side opposite* $\angle A$,
and AC is the *side adjacent to* $\angle A$.

That is,

$$\textit{tangent of } \angle A=\frac{\textit{opposite side}}{\textit{adjacent side}}, \quad \text{or} \quad \tan A=\frac{\textit{opp.}}{\textit{adj.}}.$$



200. Making a Table of Tangents.* With the aid of a protractor, lay off on squared paper angles of 10°, 20°, 30°, and so on, all having the same vertex A, and a common side AC, as shown in the figure. If the length of AC is 10, the tangent ratio of each angle can readily be found by measuring BC and dividing by 10. In this manner we may obtain, from the angles laid off, a "table of tangents" such as the following:

ANGLE	BC	$\frac{BC}{AC}$	TAN A
10°	1.8	$\frac{1.8}{10}$	0.18
20°	3.6	$\frac{3.6}{10}$	0.36
30°	5.8	$\frac{5.8}{10}$	0.58
40°	8.4	$\frac{8.4}{10}$	0.84

In this table the values of the tangents are correct to two decimal places. By more accurate methods it is possible to obtain tables containing three or more decimal places. The table on page 413 gives the tangents of angles from 1° to 90° to four decimal places. Thus, to find tan 47° from this table we first look in the column headed "Angle" for 47°. Opposite this number, on the same horizontal line, we find, in the column headed "Tangent," the decimal 1.0724. That is, tan 47° = 1.0724, to the nearest 0.0001.

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* The class may be divided into groups, each group completing by measurement a definite portion of a table of tangents.

TABLE OF SINES, COSINES, AND TANGENTS

Angle	Sine $\left(\frac{opp.}{hyp.}\right)$	Cosine $\left(\frac{adj.}{hyp.}\right)$	Tangent $\left(\frac{opp.}{adj.}\right)$	Angle	Sine $\left(\frac{opp.}{hyp.}\right)$	Cosine $\left(\frac{adj.}{hyp.}\right)$	Tangent $\left(\frac{opp.}{adj.}\right)$
1°	.0175	.9998	.0175	46°	.7193	.6947	1.0355
2°	.0349	.9994	.0349	47°	.7314	.6820	1.0724
3°	.0523	.9986	.0524	48°	.7431	.6691	1.1106
4°	.0698	.9976	.0699	49°	.7547	.6561	1.1504
5°	.0872	.9962	.0875	50°	.7660	.6428	1.1918
6°	.1045	.9945	.1051	51°	.7771	.6293	1.2349
7°	.1219	.9925	.1228	52°	.7880	.6157	1.2799
8°	.1392	.9903	.1405	53°	.7986	.6018	1.3270
9°	.1564	.9877	.1584	54°	.8090	.5878	1.3764
10°	.1736	.9848	.1763	55°	.8192	.5736	1.4281
11°	.1908	.9816	.1944	56°	.8290	.5592	1.4826
12°	.2079	.9781	.2126	57°	.8387	.5446	1.5399
13°	.2250	.9744	.2309	58°	.8480	.5299	1.6003
14°	.2419	.9703	.2493	59°	.8572	.5150	1.6643
15°	.2588	.9659	.2679	60°	.8660	.5000	1.7321
16°	.2756	.9613	.2867	61°	.8746	.4848	1.8040
17°	.2924	.9563	.3057	62°	.8829	.4695	1.8807
18°	.3090	.9511	.3249	63°	.8910	.4540	1.9626
19°	.3256	.9455	.3443	64°	.8988	.4384	2.0503
20°	.3420	.9397	.3640	65°	.9063	.4226	2.1445
21°	.3584	.9336	.3839	66°	.9135	.4067	2.2460
22°	.3746	.9272	.4040	67°	.9205	.3907	2.3559
23°	.3907	.9205	.4245	68°	.9272	.3746	2.4751
24°	.4067	.9135	.4452	69°	.9336	.3584	2.6051
25°	.4226	.9063	.4663	70°	.9397	.3420	2.7475
26°	.4384	.8988	.4877	71°	.9455	.3256	2.9042
27°	.4540	.8910	.5095	72°	.9511	.3090	3.0777
28°	.4695	.8829	.5317	73°	.9563	.2924	3.2709
29°	.4848	.8746	.5543	74°	.9613	.2756	3.4874
30°	.5000	.8660	.5774	75°	.9659	.2588	3.7321
31°	.5150	.8572	.6009	76°	.9703	.2419	4.0108
32°	.5299	.8480	.6249	77°	.9744	.2250	4.3315
33°	.5446	.8387	.6494	78°	.9781	.2079	4.7046
34°	.5592	.8290	.6745	79°	.9816	.1908	5.1446
35°	.5736	.8192	.7002	80°	.9848	.1736	5.6713
36°	.5878	.8090	.7265	81°	.9877	.1564	6.3138
37°	.6018	.7986	.7536	82°	.9903	.1392	7.1154
38°	.6157	.7880	.7813	83°	.9925	.1219	8.1443
39°	.6293	.7771	.8098	84°	.9945	.1045	9.5144
40°	.6428	.7660	.8391	85°	.9962	.0872	11.4301
41°	.6561	.7547	.8693	86°	.9976	.0698	14.3007
42°	.6691	.7431	.9004	87°	.9986	.0523	19.0811
43°	.6820	.7314	.9325	88°	.9994	.0349	28.6363
44°	.6947	.7193	.9657	89°	.9998	.0175	57.2900
45°	.7071	.7071	1.0000	90°	1.0000	.0000	

Exercises. *Tangents*

1. What is the height of a flagpole if the angle of elevation of its top, at a point 60 feet from the foot of the flagpole, is 44°?

Solution. Let h denote the height of the flagpole. Then we have

(1) $\frac{h}{60} = \tan 44°.$

That is, since $\tan 44° = 0.9657$,

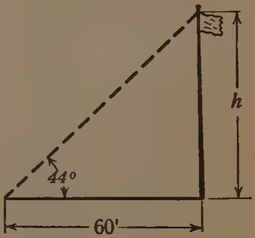
(2) $\frac{h}{60} = 0.9657.$

Hence we have

(3) $h = 60 \times 0.9657 = 57.942.$

That is, to the nearest 0.1 foot,

$h = 57.9$ feet.



Checking the Solution. In all trigonometric problems it is advisable to make, first, a scale drawing showing the given measurements, and then to find by measurement the approximate value of the required unknown parts of the figure. Since an angle of elevation is measured with a transit from a point above the ground, the value of h in this problem must be corrected by adding the height of the transit.

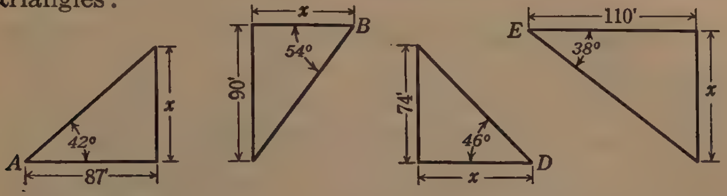
2. In Exercise 1, what is the height of the flagpole if the angle of elevation at a point 70 feet from the flagpole is 42°, the transit being 5 feet above the ground?

3. Find in the table the value of (1) tan 10°; (2) tan 24°; (3) tan 35°; (4) tan 45°; (5) tan 64°; (6) tan 79°.

4. From the table find the angles whose tangents have the following values: (1) 0.1944; (2) 0.3443; (3) 0.4877; (4) 0.7265; (5) 0.9004; (6) 1.6643.

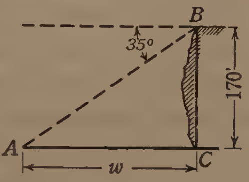
5. At a point 120 feet from the base of a tower, the angle of elevation of the top of the tower is 51°. What is the height of the tower?

6. Find the value of x in each of the following right triangles:



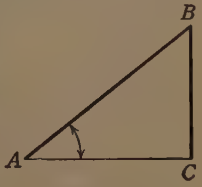
7. From the top of a cliff on one bank of a river the angle of depression of a point on the opposite bank is 35°. If the cliff is 170 feet high, how wide is the river?

SUGGESTION. Observe that in this case $\angle ABC = 90° - 35° = 55°$. Hence we can find w by using tan 55°.



8. From the top of a lighthouse which rises 110 feet above the water, the angle of depression of a boat is observed to be 3°. How far from the foot of the lighthouse is the boat?

201. Sine of an Angle. In the right triangle ABC there are other ratios, besides the tangent ratio, for $\angle A$ that are constant. For example, *the ratio* $\frac{BC}{AB}$ *does not depend on the size of the triangle, but only on the size of* $\angle A$. This ratio is called the *sine of* $\angle A$, which is abbreviated by writing "sin A." That is,

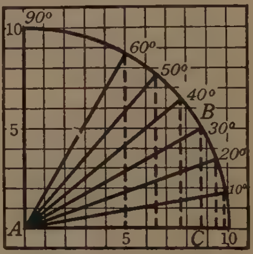


$$\textit{sine of } \angle A = \frac{\textit{opposite side}}{\textit{hypotenuse}}, \quad \text{or} \quad \sin A = \frac{\textit{opp.}}{\textit{hyp.}}.$$

A table of sines can be constructed, as in the case of tangents, by means of a sheet of squared paper. Such a table, correct to four places, is given on page 413. The meaning and use of such a table will become clear from the following exercises:

Exercises. *Sines*

1. Using the method explained on page 412, lay off a set of angles on squared paper, as shown in the diagram. With A as a center, draw an arc with the radius 10. Then the sine of $30° = \frac{BC}{AB} = \frac{5}{10}$, or 0.5.

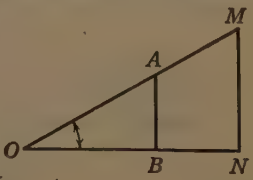


In a similar manner, find the approximate values for sin 10°, sin 20°, sin 40°, · · ·, sin 80°, and compare your results with those given in the table of sines on page 413.

2. From the table find the value of (1) sin 11°; (2) sin 24°; (3) sin 40°; (4) sin 45°; (5) sin 62°; (6) sin 76°.

3. Using the table, find the angles whose sines have the following values: (1) 0.2419; (2) 0.4384; (3) 0.5736; (4) 0.7547; (5) 0.9205; (6) 0.9945.

4. In the figure find AB if $OA = 150$ feet and $\angle O = 32°$.



Solution. (1) $\frac{AB}{OA} = \sin 32°$.

That is, since $\sin 32° = 0.5299$,

(2) $\frac{AB}{150} = 0.5299$.

Hence we have

(3) $AB = 150 \times 0.5299 = 79.485$.

That is, to the nearest 0.1 foot, $AB = 79.5$ feet.

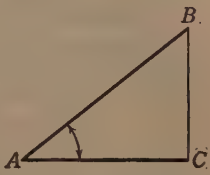
5. Find MN, in the figure above, if $OM = 230$ feet and $\angle O = 40°$.

6. Find OM if $MN = 400$ feet and $\angle O = 27°$.

7. The length of a kite string is 300 feet and it makes an angle of 65° with the ground. What is the height of the kite, assuming that the string is stretched straight?

8. An observer in an airplane, at a height of 2000 feet, finds that the angle of depression of a station on the ground is 25°. How far is the airplane from the station?

202. Cosine of an Angle. In the right triangle ABC, there is a third ratio, $\frac{AC}{AB}$, which is constant for $\angle A$. This ratio is called the *cosine of* $\angle A$, which in brief form is written "cos A." That is,

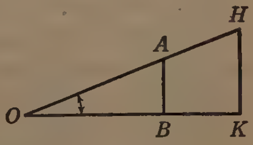


$$\textit{cosine of } \angle A = \frac{\textit{adjacent side}}{\textit{hypotenuse}}, \quad \text{or} \quad \cos A = \frac{\textit{adj.}}{\textit{hyp.}}.$$

A table of cosines, correct to four places, is given on page 413. The meaning and use of such a table will become clear from the following exercises:

Exercises. *Cosines*

1. In the figure, if $OA = 200$ feet and $\angle O = 41°$, find OB.



Solution. (1) $\frac{OB}{OA} = \cos 41°$.

That is, since $\cos 41° = 0.7547$,

(2) $\frac{OB}{200} = 0.7547$.

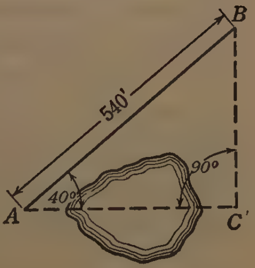
Hence we have (3) $OB = 200 \times 0.7547 = 150.94$.

That is, to the nearest 0.1 foot,

$OB = 150.9$ feet.

2. If $OH = 320$ feet and $\angle O = 53°$, in the figure above, find OK.

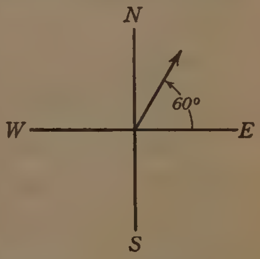
3. To find the distance from A to C, across a swamp, a surveyor used the method suggested by the diagram. With the aid of the measurements shown, find AC.



4. The diagonal of a square is 14 inches. Find the length of one side by trigonometry.

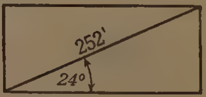
SUGGESTION. Draw a square and one of its diagonals. Observe that each acute angle in the right triangles thus formed contains 45°.

5. Suppose that a ship, after leaving the harbor, has sailed in a southeasterly direction a distance of 35 miles. How far is the ship to the east of the harbor from which it started?



6. A man walked away from his camp in a direction which was 60° north of east, at the rate of 4 miles an hour. After two hours, how far is he to the east of his camp?

7. A telegraph pole was held in position by a guy wire which reached the ground 21 feet from the foot of the pole and made an angle of 48° with the ground. What was the length of the wire?



8. The length of a path running diagonally across a rectangular field was found to be 252 feet. It formed an angle of 24° with one of the sides of the field. What were the dimensions of the field?

203. Trigonometric Ratios. Since the sine, the cosine, and the tangent are ratios used in the indirect measurement of heights and distances by means of *triangles*, they are often called *trigonometric ratios.** They have interesting relations which are of great importance in the higher branches of mathematics. Above all, they enable us to find unknown distances quickly and accurately, by a process of *computation.*

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* They are also called trigonometric *functions*, since their values depend on those of the angles with which they are associated.

The exercises which follow will serve to increase your skill in applying these ratios in the solution of trigonometric problems.

Miscellaneous Exercises. *Trigonometric Ratios*

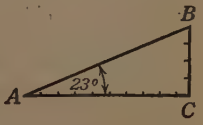
1. The angle of elevation of the top of a chimney, at a point 360 yards from its foot, is 17°. What is the height of the chimney?

2. An observer found the angle of elevation of a balloon to be 39°. The balloon was anchored directly above a point 1200 feet distant from the observer. What was the height of the balloon?

3. A wire extending from the top of a pole to a point on the ground 32 feet from the foot of the pole makes an angle of 41° with the ground. Find the length of the wire and the height of the pole.

4. Two stations, *A* and *B*, are 2 miles apart. When an airplane is directly above *A*, the angle of depression of *B* is observed to be 10°. At what height is the airplane flying?

5. Draw an angle of 23° without using a protractor.

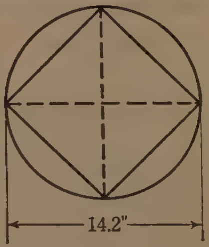


Suggestion. Since the tangent of 23°, to the nearest tenth, is 0.4, draw a right triangle ABC such that $AC = 10$ and $BC = 4$.

6. Draw an angle of 35° without using a protractor.

7. The traffic level of a new bridge was planned to be 50 feet above water level. How long must the approaches to the bridge be if their angle of elevation is to be 5° and if the street level is 10 feet above water level?

8. The diagonal of a square is 17.6 inches. Find the length of one side by trigonometry.



9. How large a square can be cut from a circular sheet of tin which has a diameter of 14.2 inches?

10. The width of a house is 32 feet. The base angles of the gable triangle are 40° each. Find (1) the height of the gable triangle; (2) the area of the gable triangle.

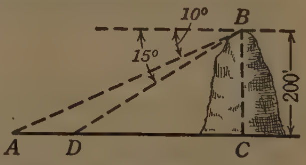
11. A steamer is traveling in a northeasterly direction at the rate of 20 miles an hour. How much farther north is it at the end of an hour?

12. From the top of a vertical cliff 90 feet high, the angle of depression of a boat was observed to be 11°. How far was the boat from the observer?

13. The length of a kite string is 120 yards, and it makes an angle of 47° with the ground. Supposing the line of the kite string to be straight, how high is the kite?

14. An observer in a lighthouse, from a window that is 96 feet above water level, finds that the angle of depression of a boat is 9°. What is the distance of the boat from the foot of the lighthouse, and how far is the observer from the boat?

15. From the top of a hill rising 200 feet above the level of a lake, the angle of depression of a canoe was found to be 15°. The angle of depression of a second canoe directly in line with the first was 10°. What was the distance AD between the canoes?



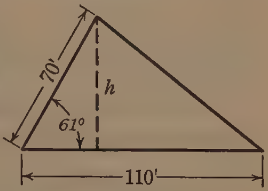
16. At a certain hour a rod 6 feet high cast a shadow 8 feet long. What was the angle of elevation of the sun at that time, to the nearest degree?

SUGGESTION. The ratio of 6 to 8, expressed decimally, is the tangent of the angle.

17. At a time when the angle of elevation of the sun is found to be 43°, a chimney casts a shadow 130 feet long. How high is the chimney?

18. Two sides of a triangular building lot are 70 feet and 110 feet. The included angle is 61°. Find the area of the lot.

SUGGESTION. Observe that $\frac{h}{70} = \sin 61°$.



19. What is the sun's altitude (angle of elevation), to the nearest degree, when a man's shadow is twice his own height? three times his height?

20. A 36-foot ladder leans against a wall and makes an angle of 69° with the ground. How far from the ground is the top of the ladder?

21. A 30-foot ladder resting against the side of a house just reaches a window when it makes an angle of 34° with the house. How far is the window above the ground?

22. The grade of a road is 5%. This means that it rises 5 feet in every 100 feet of horizontal distance. What is the angle of elevation of the road to the nearest degree?

23. What is the angle of elevation of a road, to the nearest degree, if the road rises 1.5 feet in each horizontal distance of 50 feet?

24. A straight road running from the foot of a hill to its top has a length of 1200 feet. If the road has a uniform angle of elevation of 6°, how high is the hill?

204. Interpolation. When the angles involved in a trigonometric problem are measured in degrees, minutes, and seconds, the values of the necessary trigonometric ratios are not given in a brief table such as the one shown on page 413. In that case the values of the required sines, cosines, or tangents may be found approximately by a special process which is called *interpolation.* Again, if the value of a trigonometric ratio is not given in the table, the corresponding *angle* must also be found by such a process of interpolation.

In work of this kind it is necessary to keep in mind the following important facts:

1. As the size of an acute angle increases, the corresponding values of its sine and its tangent also *increase,* while the corresponding values of its cosine *decrease.* (This fact may be verified by an inspection of the table.)
2. A small increase in the size of an angle produces, approximately, a *proportional* increase, or decrease, in the values of these trigonometric ratios.

1. *Finding a required trigonometric ratio by interpolation.*

Illustrative Example 1. Find tan 33° 40′.

Solution. Evidently, tan 33° 40′ lies between tan 33° and tan 34°. Also, since there are 60′ in one degree, $40' = \frac{2}{3}$ of 1°, and $33°\,40' = 33\frac{2}{3}°$. The required computation may be arranged conveniently as follows:

(1) $\tan 34° = 0.6745$
(2) $\tan 33° = \underline{0.6494}$
(3) $\text{Difference} = 0.0251$

(4) $\tan 33° = 0.6494$
(5) $\frac{2}{3}$ of $0.0251 = \underline{0.0167}$
(6) $\tan 33°\,40' = 0.6661$

NOTE. Since the tangent of an angle *increases* as the angle *increases,* we must *add the proportional part of the difference* (0.0167) to the value of tan 33°. In problems requiring interpolation, it is usually necessary to apply the rules for rounding off numbers (see pages 403–404).

Illustrative Example 2. Find $\cos 64° \, 12'$, or $\cos 64\frac{1}{5}°$.

Solution. The computation may be arranged as follows:

(1) $\cos 64° = 0.4384$
(2) $\cos 65° = \underline{0.4226}$
(3) Difference $= 0.0158$

(4) $\cos 64° = 0.4384$
(5) $\frac{1}{5}$ of $0.0158 = \underline{0.0032}$
(6) $\cos 64° \, 12' = 0.4352$

NOTE. Since the cosine of an angle *decreases* as the angle *increases*, it is necessary to *subtract the proportional part of the difference* (0.0032) from the value of cos 64°.

After you have become more skillful in using interpolation, some of the steps shown above may be performed mentally.

2. *Finding an angle corresponding to a given trigonometric ratio.*

Illustrative Example. Find $\angle A$ if $\sin A = 0.4258$.

Solution. The table shows that the given value of $\sin A$ lies between sin 25° and sin 26°. Hence we have the following computation:

(1) $\sin 26° = 0.4384$
(2) $\sin 25° = \underline{0.4226}$
(3) Difference $= 0.0158$

(4) $\sin A = 0.4258$
(5) $\sin 25° = \underline{0.4226}$
(6) Difference $= 0.0032$

(7) Since the value of $\angle A$ lies between 25° and 26°, it follows that $\angle A$ exceeds 25° by approximately $\frac{0.0032}{0.0158}$ of 1°, or $\frac{32}{158}$ of 60′, which equals 12′, to the nearest minute.

(8) Hence $\angle A = 25° + 12' = 25° \, 12'$.

NOTE. Fractional parts of a degree are often expressed *decimally*. Thus, $25° \, 12' = 25\frac{1}{5}° = 25.2°$. Similarly, $47.6° = 47\frac{3}{5}° = 47° \, 36'$.

Exercises. *Interpolation*

Using the process of interpolation explained above, find the value of each of the following trigonometric ratios:

1. tan 31° 30′.
2. tan 47° 15′.
3. tan 61.2°.
4. sin 20° 18′.
5. sin 39.8°.
6. sin 70° 45′.
7. cos 15° 6′.
8. cos 31.4°.
9. cos 65° 40′.

Find the value of $\angle A$ *to the nearest minute by using the process of interpolation:*

10. $\tan A = 0.2886$.
11. $\tan A = 0.5727$.
12. $\tan A = 1.6842$.
13. $\sin A = 0.2672$.
14. $\sin A = 0.4292$.
15. $\sin A = 0.7844$.
16. $\cos A = 0.9674$.
17. $\cos A = 0.8890$.
18. $\cos A = 0.7566$.

19. From a point 72 feet from the foot of a flagpole its angle of elevation was observed to be $41^\circ 15'$. Find the height of the flagpole.

20. The base of an isosceles triangle is 24, and each base angle contains $37^\circ 12'$. Find the altitude of the triangle and the area.

EXPLANATION. A triangle which has two equal sides is called an *isosceles triangle*. The equal sides are called the *legs* and the third side is the *base*. The altitude of an isosceles triangle divides the base into two equal parts.

21. The legs of a right triangle are 3 inches and 4 inches respectively. Find the number of degrees in each of the acute angles, to the nearest minute.

22. The base of a triangle is 14, and each of the other two sides is 20. Find (1) the size of each angle of the triangle, to the nearest minute; (2) the altitude of the triangle; (3) the area of the triangle.

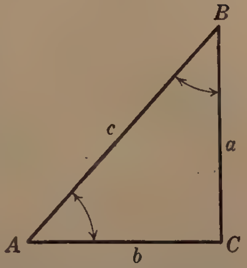
205. Summary. In this chapter you have learned

(1) The meaning and importance of indirect measurement.
(2) How to measure lines and angles directly.
(3) Important facts about triangles.
(4) How to solve trigonometric problems by scale drawing.
(5) The meaning and use of the tangent, the sine, and the cosine.
(6) How to solve trigonometric problems by computation.

Test on Chapter XV

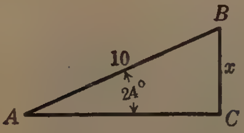
1. If the sides of a right triangle ABC are represented by a, b, and c, as shown in the diagram, complete each of the following statements:

(1) $\tan A =$ ----. (4) $\cos A =$ ----.
(2) $\sin A =$ ----. (5) $\tan B =$ ----.
(3) $\sin B =$ ----. (6) $\cos B =$ ----.

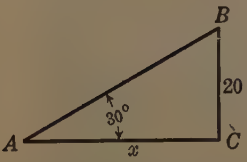


2. Using the values shown in each of the following right triangles, write the equation which enables you to find the side x:

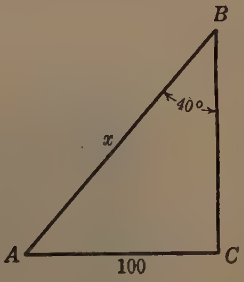
(1)



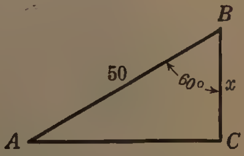
(2)



(3)



(4)



3. From the table find the number of degrees in $\angle A$ which corresponds to each of the following indicated values:

(1) $\tan A = .3839$. (3) $\cos A = .9272$. (5) $\cos A = .2924$.
(2) $\sin A = .5299$. (4) $\tan A = 1.4826$. (6) $\sin A = .9877$.

4. Draw an angle of 40° without using a protractor.

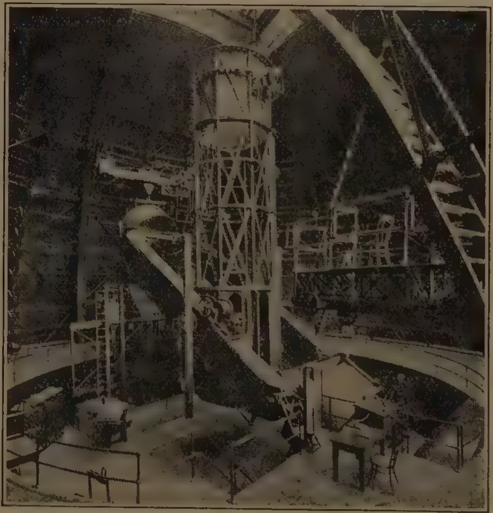
In the following problems find the required distances to the nearest 0.1 foot:

5. A straight road running from the foot of a hill to its top has a length of 980 feet. If the road has a uniform angle of elevation of 5°, how high is the hill?

6. What is the angle of elevation of the sun, to the nearest degree, if a vertical rod 5 feet long casts a shadow 8 feet long?

7. From the top of a cliff which is 120 feet high the angle of depression of a boat was observed to be 27°. How far was the boat from the observer?

8. The diagonal of a rectangle is 140 feet. It makes an angle of 31° with one of the sides. Find the dimensions of the rectangle.



The Great Reflecting Telescope at Mount Wilson Observatory in California

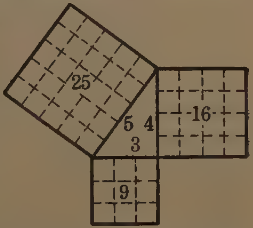
CHAPTER XVI

SQUARE ROOT AND RADICALS

1. Square Root

206. An Important Law of the Right Triangle. From the earliest times the right angle has played an important part in such practical arts as building, surveying, weaving, and the like. It is not surprising, therefore, that interesting methods of constructing right angles, and hence rectangles and squares, became known many centuries ago.

For example, to obtain a right angle, the Egyptian surveyors, or "rope stretchers," divided a rope into twelve equal sections by means of knots. They stretched this rope around three stakes so as to form a triangle whose sides contained 3, 4, and 5 of these divisions. The angle opposite the side 5 was found to be a right angle. This method was probably used in the construction of the pyramids along the Nile. It remained, however, for a Greek mathematician named Pythagoras to discover that there are many other sets of whole numbers which can also be used for such a purpose. He was able to *prove* that these sets of numbers are only special cases of a *general* relation.



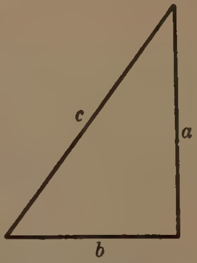
Suppose that a square is drawn on each side of a right triangle. Pythagoras proved the following important principle:

In any right triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.

This explains why the Egyptian method just described was correct.

Observe that $5^2 = 3^2 + 4^2$;

that is, $25 = 9 + 16$.



In general, if the hypotenuse of a right triangle is represented by c, and the other two sides by a and b, the above principle may be stated as follows:

$$c^2 = a^2 + b^2.$$

This relation, known as the *Pythagorean Relation* or the *Hypotenuse Rule*, is one of the most important truths of mathematics.

Observe that the formula $c^2 = a^2 + b^2$ may also be written in the form $a^2 = c^2 - b^2$ or $b^2 = c^2 - a^2$. (Explain.)

207. Applying the Hypotenuse Rule; Square Root. By means of the Hypotenuse Rule we can find the value of any side of a right triangle when the values of the other two sides are known. In solving problems of this type we shall find it necessary to review and extend our knowledge of square root.

To show that the square root of a number is to be found, the *radical sign* ($\sqrt{\ }$) is placed before the number. A bar, called the *vinculum*, is placed above the numbers which are affected by the radical sign.

Thus, $\sqrt{16} = 4$, and $\sqrt{9 + 16} = \sqrt{25} = 5$.

If $s^2 = 36$, then $s = \sqrt{36} = 6$.

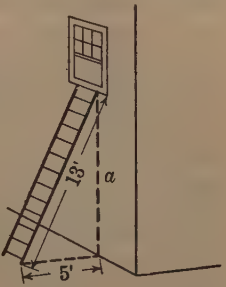
NOTE. As you have previously learned, each positive number has *two* square roots which are equal in absolute value but have opposite signs; that is, the square root of 16 may be either $+4$ or -4. These two roots are often indicated by means of the composite sign ±. Thus, $\sqrt{16} = \pm 4$, which is read "the square root of 16 is plus or minus 4." The positive root is often called the *principal* root. Only the principal root is to be used in each case, unless there is a remark to the contrary.

Illustrative Example. A ladder 13 feet long just reaches to the lower edge of a window when the foot of the ladder is 5 feet from the wall. How far is the window above the ground?

Solution. Using the formula $a^2 + b^2 = c^2$, and letting $c = 13$ and $b = 5$, we have

(1)	$a^2 + 25 = 169.$	Substitution.
(2)	$a^2 = 144.$	S_{25}.
(3)	$a = \sqrt{144} = 12.$	Why?

That is, the window is 12 feet above the ground.



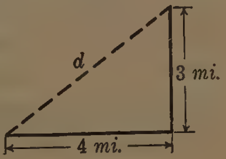
Many simple applications of the Hypotenuse Rule, such as those which appear in the exercises below, can readily be worked out by inspection. When the numbers involved are large or complicated, however, special methods of finding square roots are necessary. Some of these will be considered in the pages that follow.

Preliminary Exercises. *Using the Hypotenuse Rule*

In the following exercises let a, b, and c represent the sides of a right triangle, c being used for the hypotenuse. Find the missing side in each case.

1. $a = 5$, $b = 12$.
2. $a = 8$, $c = 10$.
3. $a = 7$, $b = 24$.
4. $b = 9$, $c = 15$.
5. $c = 17$, $a = 15$.
6. $b = 40$, $a = 30$.

7. A man walks 4 miles east and then walks 3 miles north. How far is he from his starting point?



8. How far from his starting point is a man if he travels 9 miles west and then 12 miles south?

9. By using the number pairs given below, draw the line AB on squared paper in each of the following cases, and then find its length by the Hypotenuse Rule:

(1) $A(1, 1)$; $B(7, 9)$.
(2) $A(2, 3)$; $B(10, 18)$.
(3) $A(3, 12)$; $B(8, 0)$.
(4) $A(1, 13)$; $B(17, 1)$.

10. A ladder 25 feet long just reaches the lower edge of a window. The foot of the ladder is 7 feet from the wall. How far from the ground is the window?

11. What is the longest straight line that can be drawn on a rectangular sheet of paper whose dimensions are 16 inches and 12 inches?

12. If an airplane flies 8 miles east, 10 miles south, and then 16 miles east, how far is it from its starting point?

SUGGESTION. Observe that the required distance may be made the hypotenuse of a right triangle.

208. Using a Table to find Square Roots. Since the square roots of numbers have to be found very often in practical problems, such as arise in engineering, science, mechanics, and so on, frequent use is made of *tables* of roots. The table given on page 431 makes it possible to find quickly the square root of any integer from 1 to 209, to the nearest thousandth. The top row contains the numbers from 1 to 9, and directly under each is the corresponding square root. The first column contains the numbers from 10 to 200, at intervals of 10. The second column contains the corresponding square roots of these numbers. Thus, $\sqrt{50} = 7.071$. To find the square root of an *intervening* number such as 46, find 40 in the first column. Then locate, in the same horizontal row, the number appearing in the column headed "6." The number thus located is 6.782, which is the square root of 46.

TABLE OF SQUARE ROOTS OF NUMBERS FROM 1 TO 209

n	—	1	2	3	4	5	6	7	8	9
—		1.000	1.414	1.732	2.000	2.236	2.449	2.646	2.828	3.000
10	3.162	3.317	3.464	3.606	3.742	3.873	4.000	4.123	4.243	4.359
20	4.472	4.583	4.690	4.796	4.899	5.000	5.099	5.196	5.292	5.385
30	5.477	5.568	5.657	5.745	5.831	5.916	6.000	6.083	6.164	6.245
40	6.325	6.403	6.481	6.557	6.633	6.708	6.782	6.856	6.928	7.000
50	7.071	7.141	7.211	7.280	7.348	7.416	7.483	7.550	7.616	7.681
60	7.746	7.810	7.874	7.937	8.000	8.062	8.124	8.185	8.246	8.307
70	8.367	8.426	8.485	8.544	8.602	8.660	8.718	8.775	8.832	8.888
80	8.944	9.000	9.055	9.110	9.165	9.220	9.274	9.327	9.381	9.434
90	9.487	9.539	9.592	9.644	9.695	9.747	9.798	9.849	9.899	9.950
100	10.000	10.050	10.100	10.149	10.198	10.247	10.296	10.344	10.392	10.440
110	10.488	10.536	10.583	10.630	10.677	10.724	10.770	10.817	10.863	10.909
120	10.954	11.000	11.045	11.091	11.136	11.180	11.225	11.269	11.314	11.358
130	11.402	11.446	11.489	11.533	11.576	11.619	11.662	11.705	11.747	11.790
140	11.832	11.874	11.916	11.958	12.000	12.042	12.083	12.124	12.166	12.207
150	12.247	12.288	12.329	12.369	12.410	12.450	12.490	12.530	12.570	12.610
160	12.649	12.689	12.728	12.767	12.806	12.845	12.884	12.923	12.961	13.000
170	13.038	13.077	13.115	13.153	13.191	13.229	13.266	13.304	13.342	13.379
180	13.416	13.454	13.491	13.528	13.565	13.601	13.638	13.675	13.711	13.748
190	13.784	13.820	13.856	13.892	13.928	13.964	14.000	14.036	14.071	14.107
200	14.142	14.177	14.213	14.248	14.283	14.318	14.353	14.387	14.422	14.457

Exercises. *Using a Table of Square Roots*

1. With the aid of the table given above, find the square root of each of the following numbers:

(1) 8.
(3) 75.
(5) 108.
(7) 196.
(2) 90.
(4) 12.
(6) 184.
(8) 125.

2. If a, b, and c represent the sides of a right triangle, as explained previously, find the missing side in each of the following cases:

(1) $a = 4$, $b = 5$.
(3) $c = 20$, $a = 15$.
(2) $b = 7$, $c = 10$.
(4) $a = 6$, $b = 12$.

3. On squared paper draw the line AB, in each case using the given number pairs, and then find the length of AB by the Hypotenuse Rule:

(1) $A(5, 2)$; $B(8, 3)$.
(2) $A(6, 5)$; $B(11, 2)$.
(3) $A(3, 4)$; $B(11, 6)$.
(4) $A(4, 1)$; $B(12, 11)$.

4. On squared paper draw the triangle ABC with the aid of the following number pairs, and then find its perimeter, using the Hypotenuse Rule as often as necessary:

(1) $A(4, 1)$; $B(1, 4)$; $C(1, 1)$.
(2) $A(1, 5)$; $B(9, 2)$; $C(9, 5)$.
(3) $A(2, 3)$; $B(2, 11)$; $C(4, 7)$.
(4) $A(2, 1)$; $B(11, 1)$; $C(6, 5)$.

5. Find the width of a rectangle whose diagonal is 15 and whose length is (1) 14; (2) 11; (3) 8.

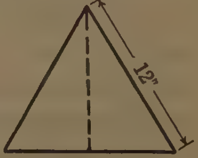
6. How long is each diagonal of a square if the length of its side is 3 feet? 7 feet? 10 feet?

7. Find the side and the area of a square whose diagonal is (1) 6 inches; (2) 10 inches; (3) 14 inches.

8. A rectangular field 11 rods long and 6 rods wide has a path running diagonally across it. How long is the path?

9. The side of an equilateral triangle is 12 inches long. Find the length of the altitude.

SUGGESTION. The altitude of an equilateral triangle divides the base into two equal parts.



10. Find the length of the altitude of an equilateral triangle which has a side of (1) 10 centimeters; (2) 8 feet; (3) 14 inches.

11. How long is one leg of an isosceles triangle if its base is 10 inches and its altitude is 13 inches?

12. The base of an isosceles triangle is 6, and each leg is 7. Find the altitude and the area.

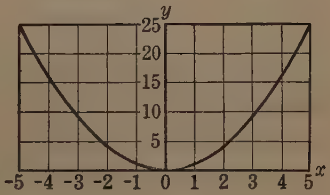
13. The altitude of an isosceles triangle is 10, and each leg is 15. Find the base and the area of the triangle.

209. Finding Square Roots by a Graph. By means of the graph of the equation $y = x^2$ it is possible to find not only the *square*, but also the *square root*, of any number coming within the range of the graph.

To draw the graph of $y = x^2$, make a table such as the one shown below. Thus, if $x = 0$, we have $y = 0^2 = 0$; if $x = -4$, $y = (-4)^2 = 16$; and so on.

If $x =$	-4	-3	-2	-1	0	1	2	3	4
then $y = x^2 =$	16	9	4	1	0	1	4	9	16

NOTE. Observe that as x is given the values 1, 2, 3, · · ·, the corresponding values of y soon become too large for convenient graphic representation if the same scale is used on both axes. Hence it is desirable to use a smaller scale on the y-axis.



To find the *square* of a number such as 3, locate that number on the horizontal axis and follow the corresponding vertical line to the curve. From the point thus located on the graph follow the horizontal line to the vertical axis. The division of the vertical axis thus located is found to be 9, the square of 3.

To find the *square root* of a number such as 16, start from 16 on the vertical axis and *reverse* the process explained in the preceding paragraph. In this case, however, the horizontal line passing through 16 meets the graph at *two* points. The corresponding division points on the horizontal axis are found to be $+4$ and -4, the two square roots of 16.

Exercises. *Finding Square Roots by a Graph*

1. Draw the graph for $y = x^2$, using successive integral values of x from -12 to $+12$.

2. Using the graph drawn in Exercise 1, find the approximate *square* of each of the following numbers: $2\frac{1}{5}$, 2.6, 3.4, 4.8.

3. With the aid of the graph drawn in Exercise 1, find to the *nearest tenth* the *square root* of each of the following numbers: 8, 13, $17\frac{1}{2}$, 20.5, 24.

4. Using the graph drawn in Exercise 1, find the approximate square root of each of the following numbers:

(1) 90. (3) 67. (5) 71. (7) 45.
(2) 56. (4) 86. (6) 137. (8) 105.

210. An Important Relation between a Square and its Square Root. The following table of squares will help you to understand a relation which is of importance in finding the square root of an arithmetic number:

$2^2 = 4$	$10^2 = 1'00$	$100^2 = 1'00'00$	$.2^2 = .04$
$9^2 = 81$	$99^2 = 98'01$	$999^2 = 99'80'01$	$.09^2 = .00'81$

Observe that when a square contains one or two figures, its square root has *one* figure. Thus, $\sqrt{4} = 2$, $\sqrt{81} = 9$, $\sqrt{.04} = .2$.

Similarly, if a square contains three or four figures, the square root has *two* figures. Thus, $\sqrt{100} = 10$, $\sqrt{9801} = 99$, $\sqrt{.0081} = .09$.

These illustrations suggest the following general relation:

If an arithmetic number which represents a square is separated into groups, or "periods," of two figures each, beginning at the decimal point, the square root of the number will have as many figures as there are groups.

Thus, $\sqrt{1'74'24} = 132$; $\sqrt{.06'25} = .25$;

$$\sqrt{4'42.26'09} = 21.03.$$

Oral Exercises. *Determining the Size of Square Roots*

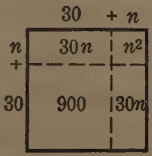
Determine the number of figures, and also the position of the decimal point, in the square root of each of the following squares:

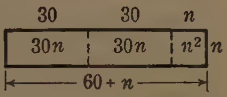
1. 49.
2. 225.
3. 106.09.
4. .0784.
5. 35,721.
6. 20.25.
7. 6.5536.
8. 10.0489.
9. .0004.
10. 2007.04.
11. 160,000.
12. .241081.

211. Finding the Square Root of Arithmetic Numbers. To find the square root of a number which is beyond the range of the table, or which cannot be found graphically with a sufficient degree of accuracy, special methods of computation are used. One of these will now be considered.

Illustrative Example 1. Find the square root of 1156.

Solution. We observe that the square root of 1156 must be a number which is greater than 30 but less than 40, since $30^2 = 900$ and $40^2 = 1600$. Accordingly, the desired root might be represented by $30 + n$. Then $(30 + n)^2$ would represent 1156, as the diagram at the right indicates.





If the large square, namely 900, is taken away from the original square 1156, the combined area of the remaining rectangles and the small square must equal 256. This remaining area may be pictured by a single rectangle, as shown above.

The problem then becomes one of finding the value of n, that is, the width of the rectangle whose area is 256 and whose length is $60 + n$. You will observe that a very close approximation may be obtained by dividing 256 by 60, which gives the approximate result 4. When this result is added to 60, we obtain 64, which is the actual length of the rectangle. And since $64 \times 4 = 256$, it is evident that when $n = 4$, $(30 + n)^2 = 1156$. That is, the square root of 1156, which is $30 + n$, becomes $30 + 4$, or 34. The number 60, which was used above to find 4, is called the *trial divisor*, while 64, the base of the rectangle, is called the *complete divisor*.

This result could have been found quickly by using the following arrangement:

$$\begin{array}{rr|l}
 & 11'56 & \underline{30 + 4 = 34} \\
 & 30^2 = 9\ 00 & \\
\text{Trial divisor, } 2 \times 30 = 60 & \multicolumn{1}{|r}{2\ 56} & \\
\text{Complete divisor, } 60 + n = 64 & \multicolumn{1}{|r}{\underline{2\ 56}} & = n(60 + n)
\end{array}$$

Hence the square root of 1156 is ± 34.

The method suggested above may be extended to the case of *any* arithmetic number.

Illustrative Example 2. Find the square root of 1239.04.

Solution. The following arrangement, in which unnecessary ciphers have been omitted, will be found more compact than the one used in Example 1. It also makes easier the location of the decimal point in the result.

$$\begin{array}{rl}
 & \underline{3\ \ 5.\ 2} \\
 & 12'39.04 \\
 & \ \ 9 \\
\text{Trial divisor, } 2 \times 30 = 60 & \multicolumn{1}{|l}{3\ 39} \\
\text{Complete divisor, } 60 + 5 = 65 & \multicolumn{1}{|l}{\underline{3\ 25}} \\
\text{Trial divisor, } 2 \times 350 = 700 & \multicolumn{1}{|r}{14\ 04} \\
\text{Complete divisor, } 700 + 2 = 702 & \multicolumn{1}{|r}{\underline{14\ 04}}
\end{array}$$

Hence the square root of 1239.04 is ± 35.2.

If the square root of a number *cannot* be found exactly, the result may be carried to as many decimal places as desired, by annexing enough ciphers to the given number to make the number of periods of two figures each, after the decimal point, equal to the number of decimal places desired in the root.

NOTE. There are many numbers whose square root cannot be found exactly. Such roots must therefore be *approximated*. Thus, the square root of 5 is $2.2360\cdots$. Hence $\sqrt{5}$, "correct to the nearest thousandth," is 2.236; whereas to the nearest hundredth it is 2.24. In general, to express the value of a root "correct to the nearest *hundredth*," it is necessary to find its value to *three* decimal places and then to *round off* this result. A similar plan is followed in the case of any other desired degree of accuracy. Thus, since $\sqrt{10} = 3.162\cdots$, the value of $\sqrt{10}$ to the nearest tenth is 3.2. Similarly, since $\sqrt{39} = 6.244\cdots$, the required root to the nearest hundredth is 6.24. Again, since $\sqrt{77} = 8.774\cdots$, the required root to the nearest hundredth is 8.77.

Exercises. *Finding Square Roots*

Find the square roots of the following numbers:

1. 1296.
2. 6889.
3. 23,409.
4. 54.76.
5. 6.6564.
6. 497.29.
7. 36,100.
8. .0841.
9. 5358.24.
10. .002809.
11. 9216.
12. 70.0569.

Find the square roots of the following numbers correct to the nearest hundredth:

13. 3.4598.
14. 12.041.
15. 10.
16. 503.
17. .45.
18. .064.
19. 3.
20. 823.7.
21. 29.7025.

Change each of the following into an equivalent decimal, and then find the principal square root correct to the nearest hundredth:

22. $\frac{3}{4}$.
23. $\frac{7}{8}$.
24. $\frac{9}{100}$.
25. $2\frac{1}{2}$.
26. $10\frac{5}{8}$.
27. $66\frac{2}{3}$.
28. $\frac{5}{4}$.
29. $\frac{63}{16}$.
30. $\frac{205}{12}$.

In each of the following exercises the result should be given to the nearest tenth:

31. Find the hypotenuse of a right triangle whose legs are 30 inches and 16 inches.

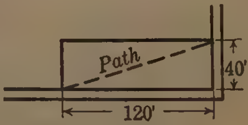
Explanation. In a right triangle the sides which form the right angle are called the *legs* of the triangle.

32. If the hypotenuse of a right triangle is 75 feet and one leg is 21 feet, find the length of the other leg.

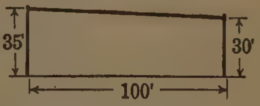
33. A 32-foot ladder is leaning against a house. If its foot is 8 feet from the house, how far above the ground is the top of the ladder?

34. A baseball diamond has the form of a square 90 feet on each side. Find the distance from third base to first base.

35. A footpath runs diagonally across a rectangular lot located at a corner. If the lot is 40 feet wide and 120 feet deep, what distance is saved by following the path rather than keeping on the walk?



36. Two poles are 100 feet apart. The poles are 35 feet and 30 feet respectively in height. How long is a wire connecting their tops?

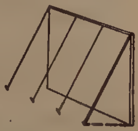


37. Can an umbrella 28 inches long be packed in the bottom of a suitcase whose inside dimensions are 24 inches and 16 inches?

38. Find the altitude and the area of an equilateral triangle whose side is 25 centimeters.

39. Find the side and the area of a square whose diagonal is (1) 5 inches; (2) 20 centimeters; (3) 14.14 feet.

40. The iron frame of a large advertising sign which is 15 feet high is held in a vertical position by four braces, as shown in the diagram at the right. If each of the braces is 17 feet long, how far from the foot of the frame must each brace be anchored?



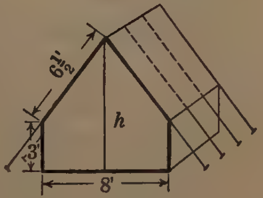
41. Draw an isosceles triangle with base b, equal sides l, and height h.

(1) If $b = 4$ inches and $l = 5$ inches, find h and also the area A.

(2) If $b = 10$ inches and $h = 7$ inches, find l.

42. The fire escape shown in the figure at the right has a platform 3 feet square which is supported by two braces each of which is $3\frac{1}{2}$ feet long. How far below the platform should the braces be attached to the building so that the platform may be level?

43. The dimensions of a tent are shown in the diagram at the right. What is the height (h) of one of the tent poles?



2. Radicals

212. Definitions. 1. If the volume of a cube is 64, we know that, since $64 = 4 \times 4 \times 4$, its edge must be 4. This relation may be indicated by writing $\sqrt[3]{64} = 4$, which is read "the *cube root* of 64 is 4." And just as the correctness of $\sqrt{9} = 3$ is proved because $3^2 = 9$, it follows that $\sqrt[3]{64} = 4$, since $4^3 = 64$.

In like manner, $\sqrt[4]{81} = 3$ is read "the *fourth root* of 81 is 3." The correctness of this statement is based on the fact that $3^4 = 81$.

In general, $\sqrt[n]{a} = b$ is read "the *nth root* of a is b." This relation implies that $b^n = a$.

2. Any expression which contains an indicated root is called a *radical expression* or simply a *radical*. As previously stated, the symbol $\sqrt{\ }$ is called the *radical sign*. The small number or letter placed above the radical sign is called the *index* of the root.

For example, in the case of $\sqrt[3]{64}$, the index of the root is 3. In the case of square roots it is customary to omit the index 2; that is, $\sqrt{9}$ has the same meaning as $\sqrt[2]{9}$.

The number or expression under the radical sign is called the *radicand*.

Thus, in the expression $\sqrt[n]{a} = b$, a is the radicand, while n is the index and b is the root.

The *degree* of a radical is determined by the index of the indicated root.

Thus, $\sqrt{2}$ and $\sqrt{ab}$ are of the *second degree*, $\sqrt[3]{5}$ is of the *third degree*, and so on.

3. Any integer, or the quotient of two integers, is called a *rational number*.

Thus, 7, 4, -3, $\frac{5}{7}$ are rational numbers.

A radical is said to be *rational* if its root can be found *exactly*.

Thus, $\sqrt{4}$, $\sqrt[3]{8}$, $\sqrt{4a^2b^2}$ are rational.

A radical is *irrational* if its root cannot be found exactly.

A *surd* is an indicated root of a *positive rational number* which cannot be found exactly.

For example, $\sqrt{2}$, $\sqrt[3]{4}$, $\sqrt[5]{7}$ are surds.

A surd of the second degree is called a *quadratic surd.* A *binomial quadratic surd* is a binomial containing at least one quadratic surd.

Thus, $3+\sqrt{5}$ and $\sqrt{7}+\sqrt{2}$ are binomial quadratic surds.

NOTE. The decimal value of a surd such as $\sqrt[3]{4}$ cannot be found by inspection. The approximate value of such a surd may, however, be obtained by drawing the graph of $y=x^3$ and using the method explained on page 433.

Oral Exercises. *Roots of Monomials*

Verify each of the following equalities:

1. $\sqrt{49}=\pm 7$.
2. $\sqrt{64\,a^2}=\pm 8\,a$.
3. $\sqrt[3]{8\,b^6}=2\,b^2$.
4. $\sqrt[3]{-125\,a^3b^9}=-5\,ab^3$.
5. $\sqrt[4]{81}=\pm 3$.
6. $\sqrt{\frac{1}{9}\,b^4}=\pm\frac{1}{3}\,b^2$.

Find the indicated root in each case:

7. $\sqrt{169}$.
8. $\sqrt[3]{27}$.
9. $\sqrt[4]{16}$.
10. $\sqrt{25\,a^2b^4}$.
11. $\sqrt[3]{-27\,a^3b^6}$.
12. $\sqrt{9\,x^2y^2}$.
13. $\sqrt[3]{-64\,c^3d^3}$.
14. $\sqrt{\frac{9}{16}\,a^4b^8}$.
15. $\sqrt[4]{a^8b^{12}}$.
16. Show that $\sqrt[3]{8}+\sqrt[4]{16}+\sqrt[5]{32}-\sqrt{36}$ equals 0.
17. Show that $(\sqrt{4})^2=\sqrt{4^2}$.
18. Show that $(\sqrt[3]{8})^3=\sqrt[3]{8^3}$.

213. Using Radicals in Multiplication. Study the following cases of multiplying radicals having the same index:

1. $\sqrt{4}\cdot\sqrt{16}=2\times 4=8=\sqrt{64}=\sqrt{4\times 16}$.
2. $\sqrt[3]{27}\cdot\sqrt[3]{8}=3\times 2=6=\sqrt[3]{216}=\sqrt[3]{27\times 8}$.
3. $\sqrt{9\,a^2}\cdot\sqrt{b^2}=3\,a\cdot b=3\,ab=\sqrt{9\,a^2b^2}=\sqrt{9\,a^2\cdot b^2}$.

From these illustrations we infer that, in general,

$$\sqrt[n]{a}\cdot\sqrt[n]{b}=\sqrt[n]{ab},\quad\text{and}\quad\sqrt[n]{ab}=\sqrt[n]{a}\cdot\sqrt[n]{b}.$$

That is, *the product of two radicals of the same degree is equivalent to the like root of the product of their radicands.* Also, *any indicated root of a product is equivalent to the product of the like roots of its factors.*

The second principle stated above is often used in transforming, or simplifying, a radical when its radicand contains a factor whose indicated root can be found.

Thus, $\sqrt{12} = \sqrt{4 \times 3} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3}$.
Similarly, $\sqrt[3]{40} = \sqrt[3]{8 \times 5} = \sqrt[3]{8} \cdot \sqrt[3]{5} = 2\sqrt[3]{5}$.
Likewise, $\sqrt{18\,a^3b} = \sqrt{9\,a^2 \cdot 2\,ab} = 3\,a\sqrt{2\,ab}$.

NOTE. If it is desired to find the *decimal* value of a numerical radical such as $\sqrt{12}$, it may be easier to find the root *directly*, rather than to obtain it from the simplified form $2\sqrt{3}$.

Exercises. *Using Radicals in Multiplication*

In each of the following cases perform the indicated multiplication and express the result in simplest form:

1. $\sqrt{3} \cdot \sqrt{5}$.
2. $\sqrt{18} \cdot \sqrt{2}$.
3. $\sqrt{7} \cdot \sqrt{10}$.
4. $\sqrt[3]{2} \cdot \sqrt[3]{4}$.
5. $\sqrt{27} \cdot \sqrt{3}$.
6. $\sqrt{2} \cdot \sqrt{3\,r}$.
7. $\sqrt{2\,n} \cdot \sqrt{8\,n}$.
8. $\sqrt[3]{9\,a} \cdot \sqrt[3]{3\,a^2}$.
9. $\sqrt{5\,b} \cdot \sqrt{5\,bc^2}$.
10. $\sqrt{16\,a} \cdot \sqrt{a^3}$.
11. $\sqrt{98\,x} \cdot \sqrt{\frac{1}{2}\,x}$.
12. $\sqrt{\frac{2}{3}} \cdot \sqrt{15\,y}$.

Transform, or simplify, each of the following radicals in the manner explained above:

13. $\sqrt{8}$.
14. $\sqrt{20}$.
15. $\sqrt{12}$.
16. $\sqrt{128}$.
17. $\sqrt{40}$.
18. $\sqrt{98}$.
19. $\sqrt[3]{16}$.
20. $\sqrt[3]{24}$.
21. $\sqrt{275}$.
22. $\sqrt{9\,a}$.
23. $\sqrt{5\,b^2}$.
24. $\sqrt[3]{27\,n^2}$.
25. $\sqrt{18\,x^3}$.
26. $\sqrt{27\,x^2y}$.
27. $\sqrt{125\,rs^2}$.
28. $\sqrt[3]{64\,a^4b^3}$.
29. $3\sqrt{90\,m^5}$.
30. $5\,a\sqrt{75\,a^2}$.

Simplify each of the following radicals and then find its principal root to the nearest hundredth, with the aid of the table on page 431.

31. $\sqrt{360}$.

Solution. $\sqrt{360} = \sqrt{36 \times 10} = 6\sqrt{10} = 6 \times 3.162 = 18.97$.

32. $\sqrt{300}$.
33. $\sqrt{450}$.
34. $\sqrt{279}$.
35. $\sqrt{408}$.
36. $\sqrt{525}$.
37. $\sqrt{333}$.
38. $\sqrt{1044}$.
39. $\sqrt{816}$.
40. $\sqrt{1275}$.

In each of the following cases perform the indicated multiplication and simplify the result:

41. $\sqrt{3} \cdot \sqrt{6}$.
42. $\sqrt{5} \cdot \sqrt{15}$.
43. $\sqrt{2} \cdot \sqrt{\frac{1}{2}}$.
44. $\sqrt{12} \cdot \sqrt{6}$.
45. $\sqrt{3} \cdot \sqrt{30}$.
46. $\sqrt[3]{4} \cdot \sqrt[3]{6}$.
47. $2\sqrt{8} \cdot \sqrt{6}$.
48. $\sqrt{3a} \cdot \sqrt{3b}$.
49. $5\sqrt{n} \cdot \sqrt{5n}$.
50. $\sqrt{5a} \cdot 2\sqrt{10ac}$.
51. $2\sqrt{2} \cdot 5\sqrt{6}$.
52. $10\sqrt{5} \cdot 6\sqrt{12}$.
53. $(2\sqrt{3})(2\sqrt{3})$.
54. $(3\sqrt{5})^2$.
55. $(5\sqrt{2a})^2$.

214. Using Radicals in Division. Observe the following simple cases of the division of radicals having the same index:

1. $\dfrac{\sqrt{64}}{\sqrt{16}} = \dfrac{8}{4} = 2 = \sqrt{4} = \sqrt{\dfrac{64}{16}}$.

2. $\dfrac{\sqrt[3]{27a^3}}{\sqrt[3]{b^3}} = \dfrac{3a}{b} = \sqrt[3]{\dfrac{27a^3}{b^3}}$.

From these illustrations we infer that, in general,

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}} \quad \text{and} \quad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

These statements may be expressed in words in a manner similar to that suggested in the preceding section.

Exercises. *Using Radicals in Division*

In each of the following cases perform the indicated division and simplify the result:

1. $\frac{\sqrt{12}}{\sqrt{3}}$.
2. $\frac{\sqrt{50}}{\sqrt{2}}$.
3. $\frac{\sqrt{32}}{\sqrt{8}}$.
4. $\frac{\sqrt[3]{16}}{\sqrt[3]{2}}$.
5. $\frac{\sqrt{a^2b}}{\sqrt{b}}$.
6. $\frac{2\sqrt{54}}{\sqrt{6}}$.
7. $\frac{6\sqrt[3]{81}}{2\sqrt[3]{3}}$.
8. $\frac{\sqrt{24a^3}}{\sqrt{6a}}$.
9. $\frac{10\sqrt{45n^2}}{5\sqrt{5}}$.
10. $8\sqrt{\frac{25}{16}}$.
11. $20\sqrt{\frac{a^2}{4b^4}}$.
12. $40\sqrt[3]{\frac{a^3}{8}}$.

215. Simplifying Radical Expressions involving Fractions. By applying the principles and processes discussed in the preceding sections of this chapter we may find the numerical value of a radical such as $\sqrt{\frac{2}{5}}$ in any one of the following three ways:

1. $\sqrt{\frac{2}{5}} = \sqrt{.4} = .632$, to the nearest .001.

2. $\sqrt{\frac{2}{5}} = \frac{\sqrt{2}}{\sqrt{5}} = \frac{1.414}{2.236} = .632.$

3. $\sqrt{\frac{2}{5}} = \sqrt{\frac{2}{5} \cdot \frac{5}{5}} = \sqrt{\frac{10}{25}} = \frac{\sqrt{10}}{5} = \frac{3.162}{5} = .632.$

In the third method the numerator and the denominator of the original fraction were multiplied by a number which made it possible to find the indicated root of the denominator. This process of making the denominator a rational number is called *rationalizing the denominator*. Observe that this method enables us to obtain the desired decimal value of the given radical very readily, since it requires merely the division of a decimal by an integer. Hence a radical such as $\sqrt{\frac{2}{5}}$ is usually changed to the simplified form $\frac{\sqrt{10}}{5}$, as just shown.

A monomial radical expression which occurs in the denominator of a fraction may always be rationalized by applying the Fundamental Principle of Fractions.

A simple illustration of this fact is given below:

$$\frac{12}{\sqrt{3}} = \frac{12 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} = \frac{12\sqrt{3}}{\sqrt{9}} = \frac{12\sqrt{3}}{3} = 4\sqrt{3}.$$

To *simplify* a radical, therefore, means to change it into an equivalent radical such that (1) *the radicand is an integer,* (2) *the radical appears only in the numerator and contains no factor whose indicated root can be found exactly.*

Exercises. *Simplifying Radicals containing Fractions*

Explain the process used in simplifying each of the following radicals:

1. $\sqrt{\frac{a}{b}} = \sqrt{\frac{ab}{b^2}} = \frac{\sqrt{ab}}{\sqrt{b^2}} = \frac{\sqrt{ab}}{b} = \frac{1}{b}\sqrt{ab}.$

2. $\frac{6}{\sqrt{2}} = \frac{6 \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} = \frac{6\sqrt{2}}{\sqrt{4}} = \frac{6\sqrt{2}}{2} = 3\sqrt{2}.$

3. $\sqrt{\frac{3x}{5y}} = \sqrt{\frac{3x \cdot 5y}{25y^2}} = \frac{1}{5y}\sqrt{15xy}.$

4. $\sqrt[3]{\frac{3}{4}} = \sqrt[3]{\frac{3 \times 2}{8}} = \frac{\sqrt[3]{6}}{2} = \frac{1}{2}\sqrt[3]{6}.$

5. $\sqrt{\frac{3a}{b^3}} = \sqrt{\frac{3ab}{b^4}} = \frac{1}{b^2}\sqrt{3ab}.$

6. $\frac{5}{\sqrt[3]{49n}} = \frac{5\sqrt[3]{7n^2}}{\sqrt[3]{343n^3}} = \frac{5}{7n}\sqrt[3]{7n^2}.$

Simplify each of the following radicals:

7. $\sqrt{\frac{2}{3}}$.
8. $\sqrt{\frac{3}{5}}$.
9. $\sqrt{\frac{8}{3}}$.
10. $\sqrt{\frac{12}{8}}$.
11. $\frac{3}{\sqrt{2}}$.
12. $\frac{5}{\sqrt{15}}$.
13. $\frac{3\sqrt{5}}{\sqrt{3}}$.
14. $5\sqrt{\frac{9}{10}}$.
15. $\sqrt{\frac{5a}{7}}$.
16. $\sqrt{\frac{8b}{12}}$.
17. $\frac{2a}{\sqrt{3}}$.
18. $2\sqrt{\frac{5}{32n}}$.
19. $\sqrt[3]{\frac{1}{3}}$.
20. $\sqrt[3]{\frac{9a}{4}}$.
21. $\frac{n}{\sqrt[3]{12}}$.
22. $\frac{x}{\sqrt[3]{2x^2}}$.

In each of the following cases rationalize the denominator, and then find the numerical value correct to the nearest hundredth:

23. $\sqrt{\frac{3}{2}}$.
24. $\sqrt{\frac{5}{8}}$.
25. $\sqrt{\frac{9}{32}}$.
26. $\sqrt{\frac{8}{10}}$.
27. $\sqrt{\frac{1}{2}}$.
28. $\frac{1}{\sqrt{3}}$.
29. $9\sqrt{\frac{2}{3}}$.
30. $\frac{2}{\sqrt{6}}$.
31. $\sqrt{\frac{5}{12}}$.
32. $\frac{6\sqrt{2}}{\sqrt{3}}$.
33. $\frac{15}{2\sqrt{2}}$.
34. $\frac{12}{5\sqrt{6}}$.
35. $\frac{18}{3\sqrt{27}}$.
36. $\frac{\sqrt{5}-\sqrt{2}}{\sqrt{2}}$.
37. $\frac{3+\sqrt{3}}{\sqrt{3}}$.
38. $\frac{10\sqrt{2}-\sqrt{7}}{2\sqrt{5}}$.

216. Adding and Subtracting Radicals. Only radicals which are of the same degree and which have the same radicands can be united by addition or subtraction. Such radicals are called *similar radicals.*

Dissimilar radicals of the same degree may sometimes be made similar by removing rational factors from the radicands, or by rationalizing denominators, as explained in previous sections.

Illustrative Example 1. Combine $5\sqrt{3} + 2\sqrt{3}$.

Solution. $5\sqrt{3} + 2\sqrt{3} = 7\sqrt{3}$.

Illustrative Example 2. Combine $\sqrt{20} - \sqrt{\frac{1}{5}}$.

Solution. $\sqrt{20} - \sqrt{\frac{1}{5}} = 2\sqrt{5} - \frac{1}{5}\sqrt{5} = \frac{9}{5}\sqrt{5}$.

Exercises. *Adding and Subtracting Radicals*

In each of the following exercises combine the radicals as indicated:

1. $3\sqrt{2} + 2\sqrt{2}$.
2. $5\sqrt{3} - \sqrt{3}$.
3. $2\sqrt{5} + \sqrt{20}$.
4. $\sqrt{8} - 6\sqrt{2}$.
5. $\sqrt{27} + \sqrt{12}$.
6. $4\sqrt{6} + \sqrt{24}$.
7. $4\sqrt{2} - \sqrt{98}$.
8. $3\sqrt{12} - \sqrt{75}$.
9. $5\sqrt{90} - 3\sqrt{40}$.
10. $5\sqrt{45} - 4\sqrt{80}$.
11. $3\sqrt{72} - 2\sqrt{50}$.
12. $2\sqrt{3} - \sqrt{\frac{1}{3}}$.
13. $\sqrt{5} + \dfrac{5}{\sqrt{5}}$.
14. $3\sqrt{18} - 4\sqrt{\frac{1}{2}}$.
15. $2\sqrt{6} - \sqrt{\frac{2}{3}}$.
16. $2\sqrt{\frac{3}{4}} + \sqrt{48}$.
17. $3\sqrt{\frac{1}{2}} + 2\sqrt{\frac{1}{8}}$.
18. $\sqrt{\frac{5}{8}} + \sqrt{\frac{8}{5}}$.
19. $4\sqrt{x} + 5\sqrt{x}$.
20. $\sqrt{8n} - \sqrt{2n}$.
21. $4\sqrt{9x} + 9\sqrt{4x}$.
22. $3\sqrt{xy} - \sqrt{4xy}$.
23. $10\sqrt{n^3} - 2n\sqrt{n}$.
24. $5\sqrt{12a^2} + \sqrt{27a^2}$.
25. $3r\sqrt{\dfrac{r}{2}} + \frac{1}{2}\sqrt{2r^3}$.

Exercises. *Polynomials containing Radicals*

Perform the indicated operations, combining terms as far as possible in each of the following exercises:

1. $5\sqrt{3} + \sqrt{3} - 2\sqrt{3}$.
2. $\sqrt{12} - \sqrt{75} + \sqrt{27}$.
3. $5\sqrt{20} - 2\sqrt{45} - 3\sqrt{5}$.
4. $6\sqrt{8} + \sqrt{32} - \sqrt{50}$.
5. $3\sqrt{n} - 7\sqrt{n} + \sqrt{n}$.
6. $\sqrt{2n} + 6\sqrt{2n} - \sqrt{8n}$.
7. $2x\sqrt{x} + \sqrt{x^3} + \sqrt{9x^3}$.
8. $4\sqrt{\frac{1}{2}} - 8\sqrt{2} + \sqrt{\frac{1}{8}}$.
9. $\sqrt{\frac{3}{4}} - 9\sqrt{\frac{1}{3}} - \frac{1}{2}\sqrt{27}$.
10. $\sqrt{40} + \sqrt{\frac{2}{5}} - 3\sqrt{10}$.

11. $10\sqrt{6}+\sqrt{54}+\sqrt{\frac{25}{6}}$.

12. $\sqrt{3a}-\sqrt{\dfrac{a}{27}}+\sqrt{27a}$.

13. $\sqrt{2}(\sqrt{3}+\sqrt{5})$.

14. $\sqrt{3}(\sqrt{3}-\sqrt{6})$.

15. $\sqrt{6}(\sqrt{2}-\sqrt{3})$.

16. $\sqrt{5}(3\sqrt{3}+\sqrt{10})$.

17. $(2+\sqrt{3})(2+\sqrt{3})$.

18. $(3-2\sqrt{5})(3+2\sqrt{5})$.

19. $(5-\sqrt{2})^2$.

20. $(3\sqrt{2}+\sqrt{3})(\sqrt{2}-3\sqrt{3})$.

21. $(\sqrt{10}+\sqrt{5})^2$.

22. $3\sqrt{2}(\sqrt{8}+2\sqrt{5})$.

23. $(4-3\sqrt{7})^2$.

24. $\left(\dfrac{2+5\sqrt{3}}{2}\right)^2$.

25. $(6-\sqrt{6})(4+3\sqrt{6})$.

26. $(\sqrt{7}+\sqrt{2})(\sqrt{7}-\sqrt{2})$.

27. If $x=5+\sqrt{3}$, find the decimal value of x^2.

SUGGESTION. First express the value of x^2 in *radical* form, and then find the *decimal* value with the aid of the table on page 431.

28. If $x=10-\sqrt{2}$, what is the value of $3x^2$?

29. If $x=3+2\sqrt{5}$, what is the value of $4x^2$?

30. If $x=1-\sqrt{6}$, what is the value of x^2-2x-5?

31. If $x=\dfrac{-1+\sqrt{33}}{4}$, what is the value of $2x^2+x$?

3. EQUATIONS INVOLVING THE USE OF RADICALS

217. Equations containing Radicals. An equation or a formula may contain one or more radicals. Equations of this kind are called *radical equations.*

For example, $\sqrt{x}=3$, $\sqrt{x^2-25}=x-7$, and $2+\sqrt{3x}=8$ are radical equations.

The solution of such an equation depends upon the following fundamental principle:

> *If both members of an equation are raised to the same power, the resulting powers are equal.*

Illustrative Example. Solve the equation $2\sqrt{3x} = 9$.

Solution.	(1) $2\sqrt{3x} = 9$.	Given.
	(2) $4 \times 3x = 81$.	Squaring.
	(3) $x = 6\frac{3}{4}$.	D_{12}.

Check.

(1) $2\sqrt{\frac{81}{4}} = 9$.

(2) $2 \times \frac{9}{2} = 9$.

(3) $9 = 9$.

NOTE. In checking such a solution only the *principal root* of each radical should be used.

Exercises. *Equations and Formulas containing Radicals*

Solve each of the following equations and check each result:

1. $\sqrt{x} = 3$.
2. $\sqrt{3x} = 12$.
3. $\sqrt{x-2} = 7$.
4. $6 = \sqrt{5x+1}$.
5. $\sqrt{x} - 3 = 4$.
6. $\sqrt{\frac{x}{3}} = 9$.
7. $\sqrt{\frac{6x}{5}} = 12$.
8. $2\sqrt{x} = 5$.
9. $3\sqrt{2x} = 15$.
10. $3 = \frac{15}{\sqrt{x}}$.
11. $4\sqrt{x+3} = 7$.
12. $\sqrt{x^2 - 8} = x + 4$.
13. $5 - x = \sqrt{x^2 + 15}$.
14. $5\sqrt{\frac{12}{x}} = 2$.
15. $\sqrt{4x^2 - 9} + 1 = 2x$.
16. $12 = \sqrt{\frac{x^2 + 92}{2}}$.

Transform and evaluate, as indicated, each of the following formulas:

17. $v = \sqrt{2gh}$. (1) Solve for h.
(2) Find h if $v = 40$ and $g = 32$.

18. $d = n\sqrt{r}$. (1) Solve for r.
(2) Find r if $d = 1.2$ and $n = .4$.

19. $d=\sqrt{\frac{3h}{2}}$. (1) Solve for h.
(2) Find h if $d=6$.

20. $w=A\sqrt{20d}$. (1) Solve for d.
(2) Find d if $w=10$ and $A=15$.

21. $t=\pi\sqrt{\frac{l}{g}}$. (1) Solve for l.
(2) Find l if $t=1$, $\pi=\frac{22}{7}$, and $g=32$.

22. $d=\sqrt[3]{\frac{80H}{N}}$. (1) Solve for H.
(2) Find H if $d=4$ and $N=150$.

23. $P=\frac{18}{\sqrt{D}}$. (1) Solve for D.
(2) Find D if $P=3$.

24. $A=\frac{.06F}{\sqrt{h}}$. (1) Solve for h.
(2) Find h if $A=4$ and $F=500$.

25. The area of a triangle whose sides are a, b, and c may be found by the formula $A=\sqrt{s(s-a)(s-b)(s-c)}$, in which $A=$ the area and $s=\frac{a+b+c}{2}$. Find the area of a triangle whose sides are 10, 17, and 21.

218. Summary. In this chapter you have learned

(1) How to find the square root of a number (a) from a table; (b) from a graph; (c) by computation.

(2) The meaning of such terms as *radical*, *index*, *radicand*, *root*, *surd*.

(3) How to find the product and the quotient of radicals of the same degree.

(4) How to rationalize the denominators of radicals involving fractions.

(5) How to find the sum or the difference of similar radicals.

(6) How to solve a radical equation.

(7) How to transform and to evaluate formulas containing radicals.

Supplementary Honor Work

Fractional Exponents. You have learned that the Rule of Exponents for Multiplication may be stated as follows:

$$a^m \cdot a^n = a^{m+n}.$$

This rule may be extended by using fractional values as exponents. It will then be seen that fractional exponents represent merely another way of indicating roots.

Thus, $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a.$

But $\sqrt{a} \cdot \sqrt{a} = \sqrt{a^2} = a.$

That is, by multiplying either $a^{\frac{1}{2}}$ or $\sqrt{a}$ by itself, we obtain a. Hence we see that $a^{\frac{1}{2}}$ *has the same meaning as* $\sqrt{a}$.

Similarly, $a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a^1 = a.$

But $\sqrt[3]{a} \cdot \sqrt[3]{a} \cdot \sqrt[3]{a} = \sqrt[3]{a^3} = a.$

That is, $a^{\frac{1}{3}}$ *has the same meaning as* $\sqrt[3]{a}$.

In general, $\mathbf{a^{\frac{1}{n}} = \sqrt[n]{a}}.$

Observe also that

$$a^{\frac{2}{3}} \cdot a^{\frac{2}{3}} \cdot a^{\frac{2}{3}} = a^{\frac{6}{3}} = a^2,$$

and that $\sqrt[3]{a^2} \cdot \sqrt[3]{a^2} \cdot \sqrt[3]{a^2} = \sqrt[3]{a^6} = a^2.$

That is, $a^{\frac{2}{3}}$ *has the same meaning as* $\sqrt[3]{a^2}$.

In general, $\mathbf{a^{\frac{m}{n}} = \sqrt[n]{a^m}}.$

Hence the *numerator* of a fractional exponent indicates the *power* to which the radicand is to be raised, while the *denominator* indicates the *root* which is to be found.

Thus, $8^{\frac{2}{3}} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4.$

NOTE. Though fractional exponents were known as early as the fourteenth century, they were first clearly explained by the English mathematician John Wallis (1616–1703).

Exercises. *Using Fractional Exponents*

Read each of the following expressions and give its value:

1. $4^{\frac{1}{2}}$.
2. $25^{\frac{1}{2}}$.
3. $36^{\frac{1}{2}}$.
4. $49^{\frac{1}{2}}$.
5. $8^{\frac{1}{3}}$.
6. $27^{\frac{1}{3}}$.
7. $16^{\frac{1}{4}}$.
8. $64^{\frac{1}{3}}$.
9. $4^{\frac{3}{2}}$.
10. $27^{\frac{2}{3}}$.
11. $100^{\frac{3}{2}}$.
12. $16^{\frac{3}{4}}$.
13. $8^{\frac{4}{3}}$.
14. $64^{\frac{2}{3}}$.
15. $81^{\frac{3}{4}}$.
16. $9^{\frac{5}{2}}$.

Simplify each of the following expressions:

17. $x^{\frac{1}{4}} \cdot x^{\frac{3}{4}}$.
18. $y^{\frac{2}{3}} \cdot y^{\frac{1}{2}}$.
19. $3\,x^{\frac{2}{3}} \cdot x^{\frac{5}{6}}$.
20. $(4\,x)^{\frac{1}{2}} \cdot x^{\frac{1}{2}}$.
21. $x^{\frac{5}{6}} \div x^{\frac{2}{3}}$.
22. $9\,x^{\frac{2}{3}} \div 3\,x^{\frac{1}{3}}$.

In each of the following cases rationalize the denominator by using fractional exponents, and simplify the result:

23. $\dfrac{1}{\sqrt[3]{a}}$. ***Solution.*** $\dfrac{1}{\sqrt[3]{a}} = \dfrac{1}{a^{\frac{1}{3}}} = \dfrac{1 \cdot a^{\frac{2}{3}}}{a^{\frac{1}{3}} \cdot a^{\frac{2}{3}}} = \dfrac{a^{\frac{2}{3}}}{a} = \dfrac{1}{a}\sqrt[3]{a^2}$.

24. $\dfrac{1}{\sqrt[3]{x^2}}$.
25. $\dfrac{1}{\sqrt[3]{2}}$.
26. $\dfrac{x}{\sqrt[3]{x^2}}$.
27. $\dfrac{6\,a}{\sqrt[3]{2\,a}}$.
28. $\dfrac{1}{\sqrt[4]{b^3}}$.
29. $\dfrac{2}{\sqrt[5]{b^3}}$.
30. $\dfrac{4}{\sqrt[3]{2}}$.
31. $\dfrac{ax}{\sqrt[4]{ax^3}}$.

Explanation. The Rule of Exponents for Division may also be extended in an interesting manner.

Thus, $a^4 \div a^4 = a^{4-4} = a^0$. But $a^4 \div a^4 = 1$. Hence a^0 has the same meaning as 1. That is,

$$a^0 = 1.$$

Similarly, $a^2 \div a^5 = a^{2-5} = a^{-3}$. But $\dfrac{a^2}{a^5} = \dfrac{1}{a^3}$. That is, $a^{-3} = \dfrac{1}{a^3}$.

In general,

$$a^{-n} = \frac{1}{a^n}.$$

32. Draw the graph of $y = 2^x$, using in succession integral values of x from $x = -4$ to $x = 4$. From the graph find the approximate value of y when $x = 3.4$.

Test on Chapter XVI

1. Find the square root of 993.932, correct to the nearest hundredth.

2. A 24-foot ladder just reaches the lower edge of a window when its foot is 6 feet from the wall of the house. How far above the ground is the window?

3. Can a straight line $12\frac{1}{4}$ inches long be drawn on a sheet of paper $7\frac{1}{2}$ inches wide and 10 inches long? Show all the work necessary to determine your answer.

In each of the following exercises select the correct answer from those given in the parentheses:

4. $5\sqrt{3} + 2\sqrt{3} = (7\sqrt{6},\ 14\sqrt{3},\ 7\sqrt{3})$.

5. $3\sqrt{2} \times \sqrt{8} = (12,\ 6\sqrt{2},\ \sqrt{48})$.

6. $2\sqrt{3} + \sqrt{27} = (2\sqrt{30},\ \sqrt{33},\ 5\sqrt{3})$.

7. $\sqrt{\frac{3}{5}} = \left(\frac{\sqrt{15}}{5},\ 5\sqrt{15},\ \frac{3\sqrt{5}}{5}\right)$.

Remove all rational factors from the radicand in each of the following:

8. $\sqrt{275}$. 9. $\sqrt[3]{40}$. 10. $\sqrt{50\,rs^2}$. 11. $5\sqrt{32\,a^3b}$.

Rationalize the denominator in each of the following expressions:

12. $\frac{\sqrt{3}}{\sqrt{5}}$. 13. $6\sqrt{\frac{2a}{3}}$. 14. $\frac{3}{\sqrt{10\,n^3}}$. 15. $\frac{20}{\sqrt[3]{4}}$.

16. Expand $(2\sqrt{5} - \sqrt{3})^2$.

17. Combine $3\sqrt{40} - \frac{\sqrt{2}}{\sqrt{5}} - \sqrt{90} + 4\sqrt{\frac{5}{2}}$.

18. Solve the formula $r = \sqrt{\frac{7A}{22}}$ for A.

CHAPTER XVII

EQUATIONS OF THE SECOND DEGREE IN ONE UNKNOWN

219. Quadratic Equations. As you have previously learned, an equation of the *second degree* is often called a *quadratic equation*. That is, a quadratic equation in one unknown contains the *second* power, but *no higher power*, of the unknown.

A quadratic equation which contains both the second power and the first power of the unknown is called a *complete quadratic*; if it contains only the second power, it is called an *incomplete quadratic*.

For example, $x^2 - 4x - 5 = 0$ and $2x^2 - 3x = 0$ are *complete quadratic equations*, while $x^2 - 25 = 0$ and $3x^2 = 45$ are *incomplete quadratic equations*. Equations of the latter type are also called *pure quadratic equations*.

220. Solving Incomplete Quadratic Equations. It was shown on page 310 how certain incomplete quadratic equations may be solved by the *method of factoring*.

For example, the equation $x^2 - 36 = 0$ may be written in the form $(x + 6)(x - 6) = 0$. Hence, if $x + 6 = 0$, $x = -6$; and if $x - 6 = 0$, $x = 6$.

Equations of this type were also introduced in Chapter XVI, in applications of the Hypotenuse Rule, the solution being obtained, however, by the *square-root method*. In applying this method we really assumed a new principle, which may now be stated as follows:

> *If the same root is taken of both members of an equation, the resulting roots are equal.*

Illustrative Example. Solve the equation $2x^2 = 80$.

Solution. (1) $2x^2 = 80$.	Given.
(2) $x^2 = 40$.	D_2.
(3) $x = \pm\sqrt{40}$.	Square root.
(4) $x = \pm 2\sqrt{10}$.	Why?

In decimal form, since $\sqrt{10} = 3.162$, the roots, to the nearest 0.01, are ± 6.32.

Check. If $x = 2\sqrt{10}$, we have	If $x = -2\sqrt{10}$, we have
$2(2\sqrt{10})^2 = 80,$	$2(-2\sqrt{10})^2 = 80,$
or $2(40) = 80,$	or $2(40) = 80,$
or $80 = 80.$	or $80 = 80.$

NOTE. In the solution of an equation of this type both the positive and negative roots should be retained, unless the equation is based on a verbal problem in which the negative root would be meaningless. If the roots are *irrational*, it is customary in applied problems to give each result in *decimal form*, to the required degree of accuracy. For purposes of checking, however, it is best to use the *radical* form, as shown above.

Exercises. *Incomplete Quadratic Equations*

Solve each of the following equations:

1. $x^2 = 81$.
2. $2x^2 = 50$.
3. $x^2 = 27$.
4. $5x^2 = 100$.
5. $x^2 - 3 = 7$.
6. $3x^2 + 5 = 8$.
7. $6x^2 - 1 = 95$.
8. $8x^2 - 3 = 5x^2 + 144$.
9. $\frac{x^2}{5} = 20$.
10. $\frac{x^2 - 4}{3} = 36$.
11. $\frac{x^2}{2} - x = \frac{4 - 5x}{5}$.
12. $4x - \frac{1}{3} = \frac{x(24 - x)}{6}$.
13. $\frac{nx + a}{n + ax} = \frac{px + q}{p + qx}$.

Transform and evaluate, as indicated, each of the following formulas:

14. $A = s^2$. (1) Solve for s.
(2) Find s if $A = 125$.

15. $W = A^2R$. (1) Solve for A.
(2) Find A if $W = 1620$ and $R = 5$.

16. $s = \frac{1}{2}\, at^2$. (1) Solve for t.
(2) Find t if $s = 80$ and $a = 32$.

17. $H = \dfrac{nd^2}{2.5}$. (1) Solve for d.
(2) Find d if $H = 25.35$ and $n = 6$.

18. $A = \pi r^2$. (1) Solve for r.
(2) Find r if $A = 88$.

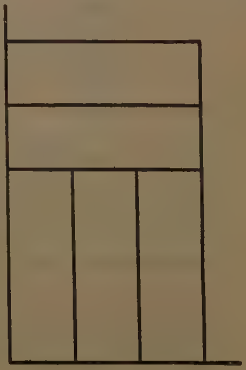
19. $S = 6\, e^2$. (1) Solve for e.
(2) Find e if $S = 216$.

20. $S = 4\, \pi r^2$. (1) Solve for r.
(2) Find r if $S = 300\, \pi$.

21. $V = \pi r^2 h$. (1) Solve for r.
(2) Find r if $V = 462$ and $h = 12$.

22. One leg of a right triangle is twice the other, and the hypotenuse is 10 inches. Find the length of each leg.

23. A farmer decided to sell one acre of his land for building lots. He wished to divide it into five lots of equal size, arranged as shown in the diagram, the length of each lot being three times the width. Find the dimensions of each lot to the nearest 0.1 of a foot. (One acre contains 43,560 square feet.)



24. A window contains four equal rectangular panes of glass, each of which is twice as long as it is wide. If 18 square feet of glass is required for the window, find the dimensions of each pane.

25. What radius must be used to obtain a circle whose area shall be 25 square feet? Give the result correct to the nearest 0.01 of a foot.

26. If a 1-gallon cylindrical paint can is to be made 7 inches high, what must be the diameter of the can, correct to the nearest 0.1 of an inch? (One gallon contains 231 cubic inches.)

221. Solving Quadratic Equations Graphically. In Chapter X it was shown how we obtain the graph of a linear equation such as $y = 3x + 5$. This method may readily be extended to equations containing *higher* powers of the unknowns. For example, consider the equation

$$y = x^2 - 2x - 3.$$

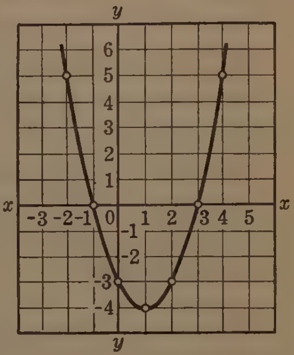
If $x = 0$, we have $y = 0^2 - 2 \cdot 0 - 3 = -3$; if $x = -2$, $y = (-2)^2 - 2(-2) - 3 = 5$; and so on, as suggested in the following table:

If $x =$	-2	-1	0	1	2	3	4
then $y = x^2 - 2x - 3 =$	5	0	-3	-4	-3	0	5

By plotting the points corresponding to these pairs of numbers on squared paper, and connecting them by a smooth curve, we obtain the graph shown in the diagram on the following page.

Observe that this curved line, which is the graph of $y = x^2 - 2x - 3$, cuts the x-axis at two points, namely, at the points -1 and 3.

Now, it is evident that the y value of any point on the x-axis is 0. Hence, at the points where the graph cuts the x-axis, $y = 0$. That is, at these points the equation $y = x^2 - 2x - 3$ becomes $0 = x^2 - 2x - 3$. In other words, since the expression $x^2 - 2x - 3$ is 0 for the values $x = -1$ and $x = 3$, we see that -1 and 3 are the *roots* of the equation $x^2 - 2x - 3 = 0$. We have thus been able to solve this equation graphically.



This graphic method of solving a quadratic equation is not always convenient, especially when the roots are fractions or decimals. It illustrates very clearly, however, the fundamental fact that *every quadratic equation has two roots.*

In solving a complete quadratic equation by the graphic method, we always obtain a curve of the kind shown above. This type of curve is called a *parabola.* There are many important applications of the parabola in engineering, industry, and science.

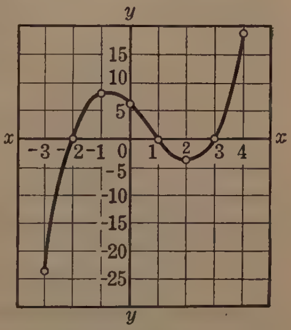
Exercises. *Graphic Solution*

Solve each of the following equations graphically, and check each root by substitution:

1. $x^2 - 4x + 3 = 0.$
2. $x^2 + 2x - 3 = 0.$
3. $x^2 + 4x - 5 = 0.$
4. $x^2 - 4x - 5 = 0.$
5. $x^2 - 2x = 0.$
6. $x^2 + 6x = 0.$
7. $x^2 - 4 = 0.$
8. $x^2 - x - 12 = 0.$
9. $x^2 + x - 20 = 0.$
10. $2x^2 - 7x + 3 = 0.$
11. $2x^2 + 3x - 20 = 0.$
12. $5x^2 + 4x - 12 = 0.$
13. $5x^2 + 3x - 36 = 0.$
14. $4x^2 - 4x - 15 = 0$

Solve each of the following equations graphically, finding each root to the nearest tenth:

15. $x^2 - 2x - 2 = 0.$
16. $x^2 - 4x - 1 = 0.$
17. $x^2 - 6x + 7 = 0.$
18. $2x^2 - 6x + 1 = 0.$
19. $4x^2 - 20x + 21 = 0.$
20. $4x^2 - 12x + 7 = 0.$



EXPLANATION. The method used in solving quadratic equations graphically can readily be extended to the case of equations that are of *higher* degree than the second.

Thus, in order to solve the equation $x^3 - 2x^2 - 5x + 6 = 0$, we draw the graph of the equation $y = x^3 - 2x^2 - 5x + 6$. This graph cuts the x-axis at the *three* points shown in the diagram. Hence $x = -2$, $x = 1$, and $x = 3$ are the three roots of the given equation.

Solve each of the following equations graphically, using the method suggested above:

21. $x^3 - 2x^2 - x + 2 = 0.$
22. $x^3 - x^2 - 6x = 0.$
23. $x^3 - 9x^2 + 23x - 15 = 0.$
24. $x^3 - 4x^2 - 11x + 30 = 0.$
25. $x^3 - 2x^2 - 35x = 0.$
26. $4x^3 - 20x^2 + 13x + 12 = 0.$

222. Solving Complete Quadratic Equations Algebraically. Certain complete quadratic equations may readily be solved by factoring (see page 310). A second method of solving such equations, which may be used in *all* cases, is based on the formula $(a + b)^2 = a^2 + 2ab + b^2$. A review of this formula and of the diagram which pictures it is therefore suggested at this point (see page 299). The following exercises will serve to explain the application of this formula in the solution of quadratic equations.

Illustrative Example 1. Solve the quadratic equation

$$x^2 + 12x + 36 = 49.$$

Solution. The left member of this equation may evidently be written in the form $(x + 6)^2$. Then we have

$$(x + 6)^2 = 49.$$

Taking the square root of both members, we have

$$x + 6 = \pm 7.$$

If $x + 6 = 7$, $x = 1$;

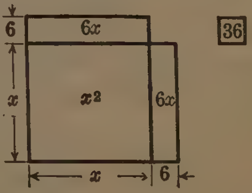
and if $x + 6 = -7$, $x = -13$.

Hence 1 and -13 are the required roots, which may be checked by substitution in the given equation.

Illustrative Example 2. Solve the equation $x^2 + 12x = 13$.

Solution. The left member of this equation, namely $x^2 + 12x$, may be pictured as shown in the adjoining diagram. Observe that the first term, x^2, is represented by the square whose side is x, while the second term, $12x$, represents the *sum* of the areas of the two rectangles, each having the sides 6 and x.

If we add to this figure, in the corner, a square having the area 6^2, or 36, the area of the resulting figure is $x^2 + 12x + 36$. That is, the "completed" square is a picture of $(x + 6)^2$.



Hence, by adding 36 to both members of the given equation, we obtain $x^2 + 12x + 36 = 49$. The left member of this equation is a *trinomial square*. We can therefore find the square root of both members and complete the solution as shown in the preceding example.

In brief form, the solution may be arranged as follows:

(1)	$x^2 + 12x = 13.$	Given.
(2)	$x^2 + 12x + 36 = 49.$	Completing the square.
(3)	$x + 6 = \pm 7.$	Square root.
(4)	If $x + 6 = +7$, $x = 1.$	S_6.
(5)	If $x + 6 = -7$, $x = -13.$	S_6.

NOTE. Observe that we found the missing term 36 as follows: (1) By dividing $12x$ by 2, we obtained $6x$, which represents the area of one of the rectangles; (2) by dividing $6x$ by x, the side of the square x^2, we obtained 6, the width of the rectangle, which is also the side of the missing square; (3) by squaring 6, we obtained the area necessary to complete the square. This is equivalent to saying that *the missing term is the square of half the coefficient of x.*

This method of solving a complete quadratic equation is called the *method of completing the square.* It may be summarized as follows:

1. *Reduce the given equation, if necessary, to the form* $x^2 + px = q$.

Thus, if $3x^2 + 6x = 9$, we have $x^2 + 2x = 3$, by division.

2. *Add to each member the square of half the coefficient of x, namely,* $\left(\frac{p}{2}\right)^2$.
3. *Take the square root of each member, using the ± sign before the square root of the second member.*
4. *Solve the two resulting first-degree equations.*
5. *Check the resulting roots, each being expressed in its simplest form.*

NOTE. The rule stated above applies also in the case of an equation having the form $x^2 - px = q$.

Illustrative Example. Solve the equation $3x^2 - 13x = 10$.

Solution.	(1)	$3x^2 - 13x = 10$.	Given.
	(2)	$x^2 - \frac{13}{3}x = \frac{10}{3}$.	D_3.
	(3)	$x^2 - \frac{13}{3}x + (\frac{13}{6})^2 = \frac{10}{3} + \frac{169}{36} = \frac{289}{36}$.	$A_{(\frac{13}{6})^2}$.
	(4)	$x - \frac{13}{6} = \pm \frac{17}{6}$.	Square root.
	(5)	$x = \frac{13}{6} \pm \frac{17}{6}$.	$A_{\frac{13}{6}}$.
	That is,	$x = 5$ or $-\frac{2}{3}$.	

Check. If $x = 5$, we have

$$3(5)^2 - 13(5) = 10,$$

or $$75 - 65 = 10,$$

or $$10 = 10.$$

If $x = -\frac{2}{3}$, we have

$$3(-\tfrac{2}{3})^2 - 13(-\tfrac{2}{3}) = 10,$$

or $$\tfrac{4}{3} + \tfrac{26}{3} = 10,$$

or $$10 = 10.$$

Exercises. *Solution by Factoring*

Solve each of the following equations by factoring, and check the roots:

1. $x^2 - 6x + 5 = 0.$
2. $x^2 - 2x - 3 = 0.$
3. $2x^2 + 7x + 3 = 0.$
4. $2x^2 - 5x + 2 = 0.$
5. $x^2 + 2x - 8 = 0.$
6. $7x^2 - 5x = 2.$
7. $x^2 - 8x + 12 = 0.$
8. $6n^2 + n = 5.$
9. $2x^2 - 13x - 7 = 0.$
10. $5x^2 + 5x - 5 = 1 - 2x.$
11. $p^2 - 9p = 0.$
12. $y^2 = 12y - 36.$
13. $9x^2 - 6x + 4 = 6x.$
14. $2R^2 + 8R = 0.$
15. $10x^2 + 21x - 10 = 0.$
16. $9x^2 + 66x = 48.$
17. $15 = 8x^2 + 19x.$
18. $x^2 - 4nx + 3n^2 = 0.$
19. $x^2 + 5bx - 6b^2 = 0.$
20. $4x^2 - 35cx + 24c^2 = 0.$

Exercises. *Solving by Completing the Square*

In each of the following exercises supply the term which is needed to make the expression a trinomial square, and give the square root of the resulting trinomial:*

1. $x^2 + 2x + (\quad).$
2. $x^2 + 6x + (\quad).$
3. $x^2 - 4x + (\quad).$
4. $y^2 - 10y + (\quad).$
5. $n^2 - 20n + (\quad).$
6. $x^2 + 3x + (\quad).$
7. $x^2 - 5x + (\quad).$
8. $y^2 - 9y + (\quad).$
9. $x^2 + x + (\quad).$
10. $n^2 - \frac{2}{3}n + (\quad).$
11. $x^2 + \frac{4}{5}x + (\quad).$
12. $x^2 + \frac{1}{2}x + (\quad).$
13. $y^2 + \frac{15}{4}y + (\quad).$
14. $x^2 - 2ax + (\quad).$
15. $x^2 - 4cx + (\quad).$
16. $n^2 + 12bn + (\quad).$

---

*A trinomial such as $x^2 + 6x + 9$ is sometimes called a *perfect* trinomial square.

Solve each of the following equations by completing the square, and check the resulting roots. Express the results in simplest radical form:

17. $x^2 + 2x = 8.$

18. $x^2 + 6x = 16.$

19. $x^2 - 4x = 12.$

20. $x^2 + 8x - 9 = 0.$

21. $x^2 - 10x + 24 = 0.$

22. $x^2 - 2x = 1.$

23. $x^2 - 4x + 1 = 0.$

24. $y^2 + 5y = 6.$

25. $x^2 - 3x - 5 = 0.$

26. $w^2 - 9w + 12 = 0.$

27. $7x = 2 - x^2.$

28. $2R^2 - 6R = 3.$

29. $4x^2 + 8x + 1 = 0.$

30. $3n^2 - 9n + 4 = 0.$

31. $2x^2 + 5x = 3.$

32. $3d^2 + 2d - 5 = 0.$

33. $9x^2 + 12x = 1.$

34. $5x^2 - x - 1 = 0.$

35. $\frac{x}{2} = \frac{24}{x-2}.$

36. $x = \frac{5}{6} + \frac{1}{x}.$

37. $\frac{50}{x+2} + 1 = \frac{50}{x}.$

38. $x^2 - 6ax = 6a^2.$

39. $x^2 + 3bx + 2b^2 = 0.$

40. $4x^2 - 8bx - b^2 = 0.$

Solve each of the following equations by completing the square, and express all irrational roots correct to the nearest hundredth:

41. $x^2 - 2x = 11.$

42. $x^2 + 6x = 11.$

43. $x^2 = 11x + 1.$

44. $x(15 - x) = 54.$

45. $4x^2 - 12x + 7 = 0.$

46. $3x^2 - 4x - 9 = 0.$

47. $x^2 + (x+1)^2 = 145.$

48. $\frac{6}{x-1} - \frac{4}{x+1} = 3.$

49. $\frac{x}{x+2} - \frac{x+2}{x} = \frac{5}{2}.$

50. If three times a certain number is subtracted from the square of the number, the result is 54. Find the number.

51. The sum of the squares of two consecutive even numbers is 340. Find the numbers.

52. If the area of a 6-inch square is to be doubled by adding the same length to each side, by how many inches must each side be increased?

223. Solving Complete Quadratic Equations by Formula. The equation $ax^2 + bx + c = 0$, in which a, b, and c represent known numbers, is called the *standard*, or *general*, quadratic equation, since every quadratic equation may be written in that form. Hence, by solving the equation $ax^2 + bx + c = 0$, as shown below, we may obtain a *formula* which will enable us to solve *any* quadratic equation.

Subtracting c from both members of the equation

$$ax^2 + bx + c = 0,$$

we have

$$ax^2 + bx = -c.$$

Dividing by a,

$$x^2 + \frac{bx}{a} = -\frac{c}{a}.$$

Completing the square, by adding $\left(\frac{b}{2a}\right)^2$ to both members, we obtain

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

$$= \frac{b^2 - 4ac}{4a^2}.$$

Taking the square root of both sides,

we have

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}.$$

That is,

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a},$$

or

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This is called the *quadratic formula.* Its use in solving quadratic equations will be shown in the following exercises:

Illustrative Example 1. Solve the equation

$$2x^2 + 5x - 12 = 0.$$

Solution. By comparing the given equation $2x^2 + 5x - 12 = 0$ with the standard form $ax^2 + bx + c = 0$, we see that $a = 2$, $b = 5$, and $c = -12$.

Substituting these values in the formula, we have

$$x = \frac{-5 \pm \sqrt{25 - 4 \cdot 2 \cdot (-12)}}{4} = \frac{-5 \pm \sqrt{25 + 96}}{4} = \frac{-5 \pm \sqrt{121}}{4}.$$

That is, $x = \frac{-5 \pm 11}{4} = \frac{6}{4}$ or $-\frac{16}{4}$.

Hence $x = 1\frac{1}{2}$ or -4.

Check. Substitute each root in the given equation.

Illustrative Example 2. Solve the equation $3x^2 - 5x = 4$.

Solution. (1) $3x^2 - 5x - 4 = 0.$	S_4.
(2) $a = 3$, $b = -5$, $c = -4$.	Why?
(3) $x = \frac{5 \pm \sqrt{25 + 48}}{6} = \frac{5 \pm \sqrt{73}}{6}.$	Formula.
(4) $x = \frac{5 \pm 8.544}{6} = \frac{13.544}{6}$ or $\frac{-3.544}{6}.$	Decimal form.
That is, $x = 2.26$ or -0.59, to the nearest 0.01.	

Check. The radical form of each root may be checked *exactly* by substitution, while the decimal values will check only *approximately.*

Exercises. *Solution by Formula*

Solve each of the following equations by using the quadratic formula:

1. $x^2 + 2x - 3 = 0.$
2. $x^2 - 8x + 15 = 0.$
3. $x^2 + 2x - 15 = 0.$
4. $x^2 - 9x - 22 = 0.$
5. $2x^2 - 9x + 4 = 0.$
6. $3x^2 + 16x + 5 = 0.$
7. $6x^2 - 5x + 1 = 0.$
8. $5x^2 - 11x + 2 = 0.$

9. $12x^2 + x - 1 = 0$.
10. $4x^2 - 24x + 35 = 0$.
11. $12y^2 + 25y + 12 = 0$.
12. $9n^2 - 15n = 14$.
13. $6r^2 - 13r + 6 = 0$.
14. $2x^2 + 21x + 10 = 0$.
15. $3x^2 + 35x = 12$.
16. $x^2 + 19 = 20x$.
17. $x^2 - 17x + 16 = 0$.
18. $x^2 + 87 = 32x$.
19. $x^2 + 13x - 140 = 0$.
20. $y^2 + y - 210 = 0$.
21. $2x^2 + x - 105 = 0$.
22. $3x^2 + 2x - 96 = 0$.
23. $x^2 - 6x + 7 = 0$.
24. $x^2 - 8x = 12$.
25. $2x^2 - 6x - 3 = 0$.
26. $3x^2 + 8x = 2$.
27. $x^2 + 25 = 15x$.
28. $25x^2 - 10x = 11$.
29. $x^2 + 2ax - 3a^2 = 0$.
30. $y^2 + 4cy - 12c^2 = 0$.
31. $x^2 + 4rx = 5r^2$.
32. $y^2 - 6cdy = 7c^2d^2$.
33. $x^2 - dx = 3d^2$.
34. $4x^2 - 8nx + n^2 = 0$.

Practice Exercises. *Solving Complete Quadratic Equations*

Solve each of the following equations by using the most convenient method, expressing all irrational roots correct to the nearest tenth:

1. $x^2 + 5x + 4 = 0$.
2. $x^2 - 2x = 63$.
3. $2x^2 - 6x + 3 = 0$.
4. $3x^2 - 5x + 1 = 0$.
5. $x^2 + 7x + 10 = 0$.
6. $x^2 - 24 = 5x$.
7. $5x^2 + 2x - 8 = 0$.
8. $n^2 - 10n + 22 = 0$.
9. $8x^2 + 2x = 1$.
10. $4c^2 + 1 = 12c$.
11. $x^2 - 11x + 30 = 0$.
12. $x(3x - 2) = 96$.
13. $x^2 + (x + 4)^2 = (x + 8)^2$.
14. $6p^2 = 7 - 10p$.
15. $10x^2 - 9x + 2 = 0$.
16. $2x(24 - 2x) + 48x = 12$.
17. $(12 - x)(18 - 2x) = 144$.
18. $9S^2 + 3S - 2 = 0$.

19. $\frac{x(x+3)}{2}=35.$

20. $x-\frac{1}{x}=\frac{3}{2}.$

21. $n+\frac{3}{n}=6\frac{1}{2}.$

22. $\frac{x}{3}=\frac{3}{x-3}.$

23. $\frac{4}{x}+\frac{1}{2}=\frac{4}{x-2}.$

24. $\frac{x-12}{3x}-\frac{5(x-5)}{9}+3=0.$

25. $\frac{2x-3}{2}-\frac{3(x^2-1)}{4x}=\frac{2}{3}.$

26. $\frac{x-3}{x+2}-\frac{x-5}{x}=\frac{1}{8}.$

27. $\frac{75}{x}+\frac{90}{x+15}=4\frac{1}{2}.$

28. $\frac{3x}{x-4}-\frac{2x-3}{x+4}=\frac{1}{4}.$

Exercises. *Problems*

1. The sum of two numbers is 15. Their product is 56. Find the numbers.

2. The sum of the squares of two consecutive integers is 85. What are the numbers?

3. One ninth of a number equals the reciprocal of the number. Find the number.

4. The perimeter of a rectangle is 108 feet. Its area is 720 square feet. Find the length of each side.

5. A rectangular garden whose dimensions are 8 feet and 12 feet is to be enlarged to twice its present size by increasing each of its dimensions by the same amount. How many feet must be added to each dimension?

6. The sum of the reciprocals of two consecutive integers is $\frac{17}{72}$. What are the numbers?

7. A pupil was asked to divide $\frac{1}{2}$ by a certain number. Instead, he added $\frac{1}{2}$ to the same number, and yet obtained the correct result. What was the number?

8. A boy finds that he can arrange his marbles in the form of a square composed of 13 rows, each row containing 13 marbles. He can also arrange them so as to form two unequal squares, each row in the larger square containing 7 more marbles than a row in the smaller square. How many marbles does he use to form each of the squares?

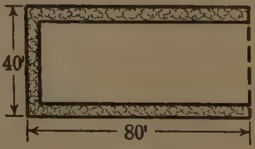
9. James was asked to divide 10 by a certain number. Instead, he divided the number by 10, thus obtaining a result which was too small by $3\frac{3}{4}$. What was the number?

10. If 3 is subtracted from the square of a certain whole number, the result exceeds nine times the number by 7. Find the number.

11. In a square the diagonal is to be 5 inches longer than the side. How long must the side be?

12. The base of a triangle is 6 feet more than its altitude, and the area is 20 square feet. Find its base and its altitude.

13. A man wishes to have a border for flowers along both sides and across the end of his back yard, the dimensions of which are 80 feet and 40 feet, as shown in the adjoining diagram. If the border is to be of uniform width and to occupy one fourth of the yard, how wide should it be?



14. Henry was asked how old he was. He said, "If you subtract 4 from the number which expresses my age in years, and square the resulting number, you obtain 100." Find Henry's age.

15. A man gave away 64 pennies to a group of children. Each child received four times as many pennies as there were children. How many pennies did each child receive?

16. Three numbers are in the ratio of 1:3:7. The sum of their squares is 531. Find the numbers.

17. A photograph 4 inches by 6 inches is to be enlarged to twice its original area. What will be the dimensions of the enlarged photograph, to the nearest hundredth of an inch?

SUGGESTION. The sides of the enlarged photograph have the same ratio as those of the original.

18. If the four numbers 2, 6, 9, and 28 are increased, respectively, by n, $2n$, $3n$, and $4n$, the resulting numbers, when taken in order, are in proportion. Find the value of n.

19. The surface (S) of a sphere of radius r is found by the formula $S = 4\pi r^2$. Find the radius of a sphere whose surface is 200π.

20. An air-mail pilot flew from Buffalo to Chicago, a distance of approximately 525 miles. He made the return trip in $1\frac{3}{4}$ hours less "actual flying time" by increasing the speed of his plane by 25 miles per hour. Find the speed at which each trip was made.

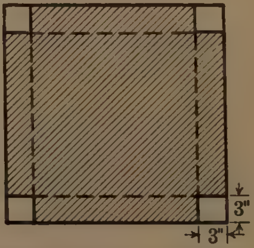
21. Kansas City and St. Louis are 200 miles apart by air, while the distance by road is 80 miles longer. An airplane and a bus leave Kansas City for St. Louis at the same time. The bus travels at a rate of 40 miles per hour less than the airplane, requires a total of $\frac{1}{2}$ hour for stops, and arrives in St. Louis 5 hours later than the plane. How long does each take to make the trip, and at what rate does each travel?

22. An artificial pond can be filled in 6 hours by two pipes when both are used at the same time. In how many hours can the pond be filled by each pipe alone if the first takes 5 hours longer than the second?

23. The number of telephone connections (C) which are possible on a switchboard with which n telephones are connected is expressed by the formula $C = \frac{n(n-1)}{2}$. If 780 connections are possible, how many telephones are there?

24. If an object is thrown vertically upward with a velocity of v feet per second, its height (s) above the ground after any number of seconds (t) is expressed by the formula $s = vt - 16t^2$. If $s = 96$ feet and $v = 80$ feet per second, what is the value of t? Are both values of t possible?

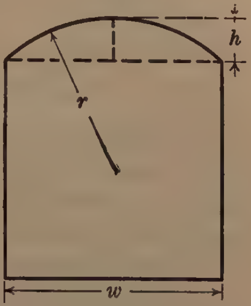
25. A drip pan is to be made from a square piece of galvanized iron by cutting out a 3-inch square from each corner and turning up the sides. How long must each side of the original square be if the pan is to contain 972 cubic inches?



26. An old Chinese arithmetic, said to have been written about 2600 B.C., contains the following puzzle problem: "In the middle of a pond 10 feet square there grew a reed. The reed projected 1 foot above the surface of the water. When blown aside by the wind, its top part reached to the mid-point of a side of the pond. How deep was the pond?" Can you solve the problem? Does it require a knowledge of quadratic equations?

27. The relation between the distance (d) a person can see and the height (h) from which the observation is made is given by the formula $d^2 = 2rh + h^2$, in which r is the radius of the earth. Solve this formula for h.

28. In a circular arch such as the one shown in the diagram, the relation between the width of the arch (w), the radius of the circle (r), and the height of the crown (h) is given by the formula $r = \frac{w^2 + 4h^2}{8h}$. Solve for h in terms of r and w.



29. A circular wading-pool 50 feet in diameter is surrounded by a walk of uniform width. If the walk contains 864 square feet, what is its width, to the nearest tenth of a foot?

30. A clothier bought a job lot of suits for \$750. He sold all but three of them for \$864, thereby making a profit of \$7 on each suit sold. What was the original cost of each suit?

31. A kitchen floor, which is 9 feet by 12 feet, is to be covered with linoleum. When taking the measurements for the amount of linoleum needed to cover the floor of the kitchen, how much allowance may be permitted in measuring the length and the width if there are to be not more than 3 square feet of waste?

224. Summary. In this chapter you have learned

(1) How to solve an incomplete quadratic equation
  (*a*) By factoring (review).
  (*b*) By the square-root method.

(2) How to solve a complete quadratic equation
  (*a*) By graph.
  (*b*) By factoring (review).
  (*c*) By completing the square.
  (*d*) By the quadratic formula.

(3) How to solve problems involving quadratic equations.

Test on Chapter XVII

1. In each of the following, select from the given equations the one which is in proper form for the method of solution specified.

(1) $x^2 - 5x = 6$, $x^2 - 5x - 6 = 0$, $x^2 - 6 = 5x$.
(To be solved by factoring.)
(2) $3x^2 + 6x - 2 = 0$, $x^2 + 2x - \frac{2}{3} = 0$, $x^2 + 2x = \frac{2}{3}$.
(To be solved by completing the square.)
(3) $2x^2 - 7x + 4 = 0$, $2x^2 - 7x = -4$, $x^2 - \frac{7}{2}x = -2$.
(To be solved by formula.)

2. Write the *quadratic formula*. In the case of the equation $3x^2 - 7x + 4 = 0$, what is the value corresponding to a? to b? to c?

3. Solve by factoring: $3x^2 - 4x = 20$.

4. Solve by completing the square: $4x^2 - 8x - 21 = 0$.

5. Solve by formula: $2x^2 - 10x + 5 = 0$.

6. Solve graphically: $x^2 + 2x - 8 = 0$, using values of x from -5 to 3.

7. Determine by substitution whether or not $3 + \sqrt{2}$ is a root of the equation $x^2 - 6x + 7 = 0$.

8. Two numbers are in the ratio of 5:7. The difference of their squares is 216. Find the numbers.

9. If each side of a square is increased by 3 inches, the area is doubled. Find the side of the original square to the nearest tenth of an inch.

10. In determining the relation between the length and the width of a well-proportioned rectangular room, architects sometimes use the formula $\frac{w}{l} = \frac{l}{w + l}$. If the length ($l$) is 20 feet, find the width (w) to the nearest tenth of a foot.

GENERAL SUMMARY AND REVIEW

1. Importance of a Final Review. A modern business firm finds it necessary to take an inventory of its stock at least once a year. In like manner, you will find it desirable at the end of your course in elementary algebra to examine the progress which you have made in your knowledge and appreciation of mathematics. The General Summary given below presents a brief outline of the processes and ideas developed in the preceding chapters. You will now understand more clearly the purpose of each of the topics you have studied. Moreover, you will realize that throughout the course considerable attention has been given to the idea of relationships between changing quantities. The general review which is to follow will serve to emphasize the fact that the study of *relationships*, or of *dependence*, is the "underlying theme of algebra."

GENERAL SUMMARY

I. *The Language and the Ideas of Algebra*

1. The use of letters in algebra.
2. Importance of mathematical shorthand.
3. The vocabulary of algebra.
4. Representing verbal statements algebraically.
5. Interpreting algebraic expressions and formulas.

II. *Fundamental Processes and Principles*

1. The four fundamental operations involving positive and negative numbers.
2. The four fundamental operations involving algebraic expressions.
3. Special products and factoring.

4. Powers and roots:
    (1) Laws of exponents in multiplication and division.
    (2) The square root of numbers.
    (3) Transformation of radicals, especially of the forms

$$\sqrt{ab} = \sqrt{a} \times \sqrt{b} \quad \text{and} \quad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \quad \text{or} \quad \frac{1}{b}\sqrt{ab}.$$

    (4) The four fundamental operations involving surds.

III. *The Equation*

1. The equation as a tool for solving problems.
2. Equations of the first degree in one unknown.
3. Equations of the first degree in two unknowns.
4. Fractional equations.
5. Equations of the second degree in one unknown.
6. Radical equations in one unknown, which lead to equations of the first degree or of the second degree.

IV. *The Formula*

1. Importance of the formula in everyday life.
2. Expressing a rule as a formula.
3. Obtaining formulas from verbal statements.
4. Obtaining a formula from a table.
5. Evaluating a formula.
6. Interpreting a formula.
7. Transforming a formula.
8. Representing a formula graphically.

V. *The Graph*

1. Interpretation of graphs used in everyday life.
2. Construction of graphs
    (1) based on statistical data; (2) based on a table.
3. Representing formulas of the types

$$y = mx + b \quad \text{and} \quad y = ax^2.$$

4. Graphic solution of two first-degree equations.
5. Graphic solution of quadratic equations.
6. Representing algebraic expressions of higher degree.
7. Graphic solution of verbal problems.

VI. *Problems*

1. The importance of problem-solving in everyday life.
2. The solution of verbal problems, such as number problems, geometric problems, business problems, work problems, motion problems, science problems, and the like.

VII. *Numerical Trigonometry*

1. Definition of the sine, cosine, and tangent.
2. Use of these ratios in solving problems involving the right triangle.

VIII. *Relationship or Dependence between Changing Quantities*

1. Importance of this idea in everyday life.
2. Methods of expressing and studying relationships:
   (1) The formula; (2) the table; (3) the equation; (4) the graph.
3. Simple cases of variation, including the study of ratio and proportion.
4. The trigonometric ratios as related to changing acute angles.

2. A Preliminary Test. The questions and exercises included in the following preliminary or "inventory" test have been made so simple that you should be able to answer them readily. Hence any difficulty which you may experience in obtaining a perfect score will serve to point out the items that should be given particular attention in your final review.

Inventory Test

PART I

In the case of each of the following statements, indicate whether it is true or false. If it is false, rewrite it in correct form.

1. The square of either a positive or a negative number is always positive.

2. Multiplying the numerator and the denominator of a fraction by 3 makes the fraction three times as large.

3. If the floor of a living room is 16 feet by 20 feet, and the corresponding dimensions of a dining room are three quarters as large, the ratio of the floor areas of these rooms is 4 : 3.

4. The square root of a positive number is less than the number.

5. One root of the equation $x^2 - 3x - 10 = 0$ is -2.

6. $\dfrac{6x^3 - 12x^2 - 3x}{-3x} = -2x^2 + 4x$.

7. $\dfrac{a-b}{a+b} \times \dfrac{a^2 - b^2}{a^2 - 2ab + b^2} = 1$.

8. $\dfrac{-6}{2+x} = \dfrac{6}{2-x}$.

9. $(2n+3)^2 = 4n^2 + 9$.

10. $(a^2b^3)^2 = a^4b^5$.

11. $(a-b)^2 = (b-a)^2$.

12. $\dfrac{3n}{4} + \dfrac{n}{6} - \dfrac{5n}{8} = \dfrac{3n}{8}$.

13. $\sqrt{a^2 + b^2} = a + b$.

14. $4 + 2\sqrt{3} = 6\sqrt{3}$.

15. $2\sqrt{\frac{3}{2}} = \sqrt{6}$.

Perform the indicated operations in each of the following:

16. Multiply $5x^2 - 2x + 4$ by $2x - 1$, and check the result.

17. Divide $12n^3 - 31n^2 + 16n - 21$ by $3n - 7$, and check the result.

Factor each of the following:

18. $7n^2 + 7$.

19. $12x^2 + 28xy - 5y^2$.

20. $25n^2 - 64$.

21. $3a^3 - 24a^2 + 48a$.

In each of the following cases perform the indicated operations:

22. $\dfrac{2n+5}{4} - \dfrac{n-2}{2} + \dfrac{4n+1}{6}$.

23. $\dfrac{r-2}{2} + \dfrac{2}{r+2} - r$.

24. $\dfrac{9a^2 - 1}{3a^2 - 7a + 2} \div \dfrac{a^2}{2a^2 - 4a}$.

25. $\dfrac{1 + \dfrac{1}{a}}{1 - \dfrac{1}{a}}$.

26. Solve and check: $8n - 5 - n = 9 + 3n - 2$.

27. Solve and check: $(x+5)^2 + 2x^2 = 14 - x(1 - 3x)$.

28. Solve and check: $\dfrac{2y-5}{2} - \dfrac{3y-3}{4} = 3$.

29. Are the following proportions correct? (Show work necessary to determine your answer.)

(1) $54 : 72 = 6 : 8$. (2) $\frac{3\,xy}{2\,x^2} = \frac{6\,y^2}{4\,y}$.

30. Solve both graphically and algebraically:

$$3\,x + y = 13.$$
$$5\,x - 4\,y = 33.$$

31. Solve each of the following equations, and check each solution:

(1) $x^2 - 9\,x = 22$. (2) $3\,x^2 + 6\,x - 1 = 0$.

32. Simplify each of the following:

(1) $3\sqrt{54}$. (2) $\sqrt{\frac{3}{5}} \times \frac{5}{\sqrt{3}}$. (3) $\sqrt{50} + \sqrt{2} - 2\sqrt{8}$.

33. Find the square root of 225.811 correct to the nearest hundredth.

PART II

Complete each of the following statements:

34. In algebra, a rule expressed in symbols is called a ----.

35. If the length of a rectangle is $2\,l$, and its width is $3\,w$, its perimeter is ----.

36. To find the amount of air space in an ordinary schoolroom, the formula ---- should be used.

37. The process of putting a number in place of a letter in a given algebraic expression or formula is called ----.

In the case of each of the following statements, indicate whether it is true or false. If it is false, rewrite it in correct form.

38. In the formula $R = \frac{C^2 + 4\,V^2}{8\,V}$, if $C = 6$ and $V = 1$, then $R = 6\frac{1}{2}$.

39. The relation between n and s, as shown in the table given below, is expressed by the formula $s = 3\,n - 3$.

If $n =$	2	3	5	12
then $s =$	9	12	18	39

40. In the formula $V = \frac{W}{T_1 - T_2}$, if all the letters represent positive integers, and W and T_1 are constant, the value of V increases as the value of T_2 decreases.

41. If $\frac{1}{N} = \frac{1}{C} + \frac{1}{R}$, then $N = \frac{CR}{C + R}$.

In the case of each of the following statements, indicate whether it is sometimes true, always true, or never true:

42. If the base of a rectangle is doubled, and also its height, the area of the rectangle is doubled.

43. A formula covers a wider range of values than its graph.

44. The graph of a formula is a straight line.

45. If the sides of a triangle are represented by a, b, and c, then $c^2 = a^2 + b^2$.

46. The area of a triangle is found by using the formula $A = bh$.

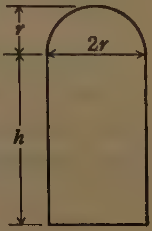
Solve each of the following as indicated:

47. A piano may be purchased on the installment plan by paying 50 dollars down and 10 dollars a month. Write a formula for determining the amount paid (p) after any number of payments (n).

48. Draw the graph representing the formula obtained in Exercise 47, using the following values of n: 0, 3, 5, 12, 18.

49. Indicate on the graph obtained in Exercise 48 how to determine the number of months required to pay for a piano which cost $240.

50. Write a formula for the total surface of a Norman window having the dimensions shown in the adjoining figure.



51. Solve the formula $V = \frac{1}{3}\pi r^2 h$ for r in terms of V, π, and h.

PART III

52. Express in algebraic language the sum of the squares of two numbers, a and b, decreased by twice their product.

53. Express in square yards the area of a rectangle that is n feet long and 15 feet wide.

54. If t tons of coal cost s dollars, indicate the cost of m tons.

Indicate which of the equations given under each of the following problems correctly states the conditions of the problem:

55. Four times the square of a certain positive number exceeds 8 times the number by 12. Find the number.

(1) $4x^2 + 8x = 12$. (2) $(4x)^2 - 12 = 8x$. (3) $4x^2 - 8x = 12$.

56. One machine can make 100 dozen buttons in 48 minutes. A smaller machine can make the same number in 1 hour and 12 minutes. How long will it take both machines together to make 100 dozen buttons?

(1) $\frac{h}{48} + \frac{h}{72} = 100$. (2) $\frac{100h}{48} + \frac{100h}{72} = 1$. (3) $\frac{h}{48} + \frac{h}{72} = 1$.

57. A merchant has tea worth 50 cents per pound and also tea worth 65 cents per pound. How many pounds of each must he use to make a mixture of 12 pounds worth 60 cents per pound?

(1) $50x + 65y = 60,$
$x + y = 12.$

(2) $x + y = 12,$
$.50x + .65y = 7.20.$

(3) $x + y = 12,$
$.50x + .65y = 720.$

Write the equations which you would use to solve the following problems:

58. At a certain hour a tree casts a shadow 27 feet long. At the same time a pole 8 feet high casts a shadow of 5 feet. Find the height of the tree.

59. A boy rides away from home in an automobile at the rate of 28 miles an hour and walks back at the rate of 4 miles an hour. The round trip requires 2 hours. How far does he ride?

60. A $10\frac{1}{2}$-foot ladder leaning against a stone wall just reaches the top of the wall and makes an angle of 64 degrees with the ground. How high is the wall?

3. General Review Exercises. The exercises which follow are arranged in groups, in the order suggested in the General Summary (page 1). Many of them are based on questions occurring in actual examinations of recent date prepared by the leading examining bodies. You may review these exercises either by groups or by selecting from the various groups any combination that may serve to remedy the difficulties pointed out by the Inventory Test.

I. The Language of Algebra

Complete correctly each of the following statements:

1. The number of miles which an airplane going at the rate of r miles per hour can fly in s hours is ____.

2. The number of pencils that can be bought for b cents, at m cents each, is ____.

3. If x bags of coal weigh y pounds, 2 bags will weigh ____.

4. If x bushels of potatoes cost n dollars, y bushels cost ____.

5. If $10x$ represents the perimeter of a rectangle, and $3x$ represents the length, the width is ____.

6. If s pounds of coffee cost t dollars, the number of pounds that can be bought for k dollars is ____.

7. A has x dollars and B has $5 less than three times as much as A. Express B's money in terms of x.

8. A baseball team played n games and lost 2. What fraction of the number of games played did the team win?

9. If it takes m minutes for a man to walk to his work, what part of the distance can he cover in 5 minutes?

10. A pupil's standings in four algebra tests were m, n, p, and q. What was his average standing for these four tests?

11. If an odd integer is represented by $2n + 1$, what expression will represent the sum of the next two larger odd integers?

12. Using only one letter, represent three numbers which are in the ratio of $3 : 4 : 5$.

13. Let s be the sum of two numbers and n be the smaller of these numbers. Express the larger number in terms of s and n and then write in algebraic form each of the following statements:

(1) The larger number exceeds the smaller number by d.

(2) The ratio of the smaller number to the larger number equals c.

(3) The smaller number multiplied by t equals the larger number divided by m.

14. The divisor is n and the quotient is q. What is the dividend d (1) if there is no remainder; (2) if there is a remainder r?

15. Jane was y years old n years ago. Represent her present age and her age 10 years hence.

16. Express p pounds and 5 ounces as ounces.

17. How many cents in d dollars and n nickels?

18. Mr. Smith paid a meat bill amounting to d dollars and c cents by giving a check for n dollars. If the value of the check exceeded the amount of the bill, what change in cents did Mr. Smith receive?

19. Jack bought d dozen oranges at c cents an orange and had 10 cents left. How many cents did he have before making the purchase?

20. Last month lemons cost n cents a dozen. Today the price per dozen is 15 cents more. How much will half a dozen lemons cost at the present rate?

21. I have d dollars. If I loan $\frac{1}{4}$ of this money for one year at 6% interest, how much interest should I receive?

22. If sugar is sold at 6 cents a pound, how many pounds can be obtained in exchange for q quarts of berries at c cents a quart?

23. An airplane travels m miles an hour. What is its rate in feet per minute?

24. John was away h hours on an automobile trip. During this time he stopped n hours at the home of a friend. If his average speed while driving was 30 miles an hour, how many miles did he drive?

II. Fundamental Processes and Principles

1. THE FOUR FUNDAMENTAL OPERATIONS

1. State the rules for the addition, the multiplication, and the division of two signed numbers having (1) like signs, (2) unlike signs.

2. Find the sum of the terms in each of the following cases:

(1) $9, -8, 15, -7$.
(2) $3xy$ and $-8xy$.
(3) $-a, -5a, 6a$.
(4) $100, -50, -70$.
(5) $3x^2y, 3xy^2, -5x^2y$.
(6) $7mn^2, -5mn^2, 4n$.

3. Subtract from $5c^2d - 3cd + 21$ each of the following expressions:

(1) $9c^2d$; (2) $-12cd$; (3) 36; (4) -22; (5) $cd + 11$.

4. State and illustrate (1) the Rule of Similar Terms; (2) the Rule of Order for Addition; (3) the Rule of Grouping for Addition.

Combine in each case:

5. $5 - 6 - 8 + 11 - 12 + 16 - 19$.

6. $9a - 7a + 8a - 7a - a + 5a + 12a$.

7. $-6x - 5 + 4x - 7 - 8x + 9 - 10x + 16$.

8. $7a^2 - c + 2a + 6c - 5a - a^2 + 7a - 4c$.

9. Given the two algebraic expressions $3x^2 - 4x + 9$ and $-10x^2 - 3x - 8$, (1) find their sum; (2) subtract the second from the first.

10. From the sum of $5a^2 - ab + b^2$ and $3ab - b^2$ subtract $3a^2 - 4ab$.

11. Add $7x - 5y + 3$ to the difference found by subtracting $x - 8y - 7$ from $3y + 5$.

12. What expression must be added to $17b^2c - 5ac^2$ to make the result $20ac^2 - 7b^2c$?

Perform the indicated operations in each case:

13. $8 + (-4) + (-12) + (-5)$.

14. $(-7) - 6 + (-10)$.

15. $-20 - (-9) - 7 + (-3)$.

16. $3 \times (-4) \times (-5) \times 2$.

17. $(+3) \cdot (+2) \cdot (-4) \cdot (-5) \cdot (-3)$.

18. $(+8) + (-6) \div (-3) - (+8) \times (-5)$.

19. State and illustrate (1) the Rule of Order for Multiplication; (2) the Rule of Grouping for Multiplication; (3) the Rule of Distribution for Multiplication; (4) the Rule of Exponents for Multiplication.

Find the product in each case:

20. $3 \times 2 \times (-5)^2$.

21. $(8ab) \cdot (-9cd)$.

22. $(-4a^2b) \cdot (-5a^4b^3)$.

23. $(3a^2b)(-5b)^2(-4a^2)$.

24. $6a^2b(-9a^3b - 5ab^2)$.

25. $-8n^3(-3n - 5n^2 + 3n^3)$.

Remove parentheses and combine similar terms:

26. $3a + 5(4a - 3b) + 2b - 7(2a + 3\text{b})$.

27. $4w(w + 3) - (7 + 6w) + (4w + 5w^2)$.

28. $-(4a - 7b + 5c) - 3(6b + 2c - 3a)$.

29. $5 - 2[7b - (10b - 4)]$.

30. $7a^2 - [(5a - 3) + 6a] + a(3 - a)$.

31. $12x^2 - [4x - 3(7x^2 + 5) - 9x]$.

32. (1) Use parentheses to indicate that $a^2 - 5a$ is to be subtracted from $8a - 6$. (2) Simplify this indicated result.

33. Subtract $-7a - 3(5b - 2c)$ from $12a - (c - 3b)$ and add the result to $9c - (8b + 3a)$.

34. What algebraic expression must be added to $3x^2 - 5x + 6$ to make the result 0?

35. From the sum of $2m - s + t$ and $s - t$, subtract $m - 3$.

36. From $11x^2 + 7x - 4$ subtract $6x^2 - 2x - 4$.

37. Simplify $22 - 3(2x - 4) + (9x + 3)$.

38. Simplify $x + 2(a - b) - (x + 3b)$.

Multiply in each case, and check the result:

39. $(3x^2 + 2x - 4)(3x^2 - 5)$.

40. $(a^2 + 2a - 3)(a^2 - 2a + 3)$.

41. $(S^2 + ST + T^2)(S - T)$.

42. $(x^2 + 2x + 4)(x - 2)$.

43. $(4ab - 5a^2 - 5b^2)(4b - 3a)$.

44. $(3c^2 + 2cd - 8d^2)(2c^2 - 5cd + 4d^2)$.

45. $(8lw + 3l^2 - 7w^2)(2l^2 + w^2 + 6lw)$.

46. State and illustrate the Rule of Exponents for Division.

Find the quotient in each case:

47. $(-16x) \div x$.

48. $25ab \div (-5)$.

49. $(-8a^3b) \div (-4ab^2)$.

50. $(-4x^3y)^2 \div 2xy$.

51. $(a^2 - 5a^4) \div a$.

52. $(9a^2b^2 - 6a^2bc) \div (-3ab)$.

53. $(18x^4 - 36x) \div (-18x)$.

54. $(a^3b - 8a^4b^5) \div (-a^2b)$.

Divide in each case, and check the result:

55. $6x^3 - 23x^2 + 29x - 12$ by $2x - 3$.

56. $10x^3 - 9x^2 + 22x - 8$ by $2x^2 - x + 4$.

57. $6x^3 + 11x^2 - 1$ by $3x + 1$.

58. $2x^3 - 7x + 5x^2 + 5$ by $2x - 1$.

59. $3x^3 - 13x^2y + 10xy^2 - 7y^3$ by $3x - y$.

60. $6m + 4m^4 - 1 - 9m^2$ by $2m^2 - 1 + 3m$.

61. $x - 2$ is an exact divisor of $x^3 - 5x^2 + 8x - 4$. What is the quotient?

62. If $x = 3y + 2$, express x^2 as a trinomial in y.

63. Simplify $(x - y)^2 - 2(x^2 + 2xy - y^2)$.

64. Simplify $(x + 3)^2 - 2(x - 3)^2 + (x + 3)(x - 3)$.

65. Simplify $(a + 2b)^2 - (2a - b)^2 - (2a + 3b)(4a - b)$.

66. Simplify $99 \times 101 - 51^2 + 18 \times 22 - 29$.

67. When $r = 5$, what is the value of $r^2 - 2r - 15$?

68. If $x = 3$, find the value of $x^2 - 7x - 9$.

69. Find the value of $a^3 + 4b^2 - 7c + 4$ when $a = 1$, $b = 2$, and $c = 3$.

70. What is the value of $15a^3 - 8a^2 + 2$ when a equals 2?

71. If $x = 2$, find the value of $5x^2$; $(5x)^2$; $(-5x)^2$; $-5(-x)^2$.

72. If $x = -3$, find by substitution the value of the expression $3x^2 + 3(-x)^2 - (-3x)^2 + (-3x^2)$.

73. Find the value of $x^3 - 5x^2 + 8x - 4$ when $x = 3$.

If $a = 1$, $b = 2$, $c = 3$, evaluate each of the following expressions:

74. $a^2 + b^2 - c^2 + 6ab - 7ac + 3bc$.

75. $(a + b)^2 + (b - c)^2 + (a - c)^2$.

76. $(a + b - c)^2 - 7(2a + 4b - 3c)^2 + 5abc$.

Factor each of the following expressions:

77. $3a^2 - 27ab$.
78. $9a^2 - 16$.
79. $x^2 - 8x + 16$.
80. $x^2 + 6x - 16$.
81. $6a^2 + 7a + 2$.
82. $2\pi rh + 2\pi r^2$.
83. $4 - 25b^2$.
84. $5x^2 - 45$.
85. $4x^2 - 28xy + 49y^2$.
86. $3x^2 - 3x - 18$.
87. $4x^2 - x - 14$.
88. $6a^2 - 9a - 6$.
89. $2a^2 - ab - b^2$.
90. $63 - 175b^2$.
91. $1.62 - 2b^2$.
92. $1 - 3a - 10a^2$.
93. $16a - 25ab^2$.
94. $7 - 8m + m^2$.

95. $16x^2 + 40xy + 25y^2$.

96. $4a^3b - 4ab^3$.

97. $2a^2b - 6ab - 20b$.

98. $n^4 - s^4$.

99. $81x^2y^2 - y^2$.

100. $3x^2 + .7x - .06$.

101. $10x^2 - 5x - 30$.

102. $25 - z^2$.

103. $81m^2 - 90m + 25$.

104. $16x^4 - 81y^4$.

105. $25x^2 - 10xy + y^2$.

106. $25a^2 + 30ab + 9b^2$.

107. $4 - .09b^2$.

108. $12a^2 - 27b^2$.

109. $9a^2 + 12ab + 4b^2$.

110. $3x^3 - 24x^2 - 60x$.

111. One factor of $2x^2 + 2x - 24$ is $x - 3$. Find the other two factors.

112. Show by division that $x - 1$ is a factor of $x^3 - 2x^2 + 2x - 1$.

113. Find the dividend when the quotient is $x - 1$, the remainder being 2 and the divisor being $3x - 1$.

EXPLANATION. It is sometimes necessary to *inclose* certain terms of an algebraic expression in parentheses. This process is merely the reverse of that of *removing* parentheses. It is obvious, therefore, that *when terms are inclosed in parentheses preceded by the minus sign, we must change the signs of the terms which are inclosed.*

Thus, $a - b + c = a - 1(+b - c) = a - (b - c)$.

Also, $x + y + z - 2a + 4b = (x + y + z) - 2(a - 2b)$.

Rewrite each of the following expressions, inclosing the x terms in one pair of parentheses and the y terms in another pair of parentheses:

114. $mx + nx - cy + dy$.

115. $rx - 5x - py - qy$.

116. $x^3 + 3x^2 - 4y^2 + ay$.

117. $4x^3 - cx^2 - ay^3 + 3y$.

118. $n^2x^3 - nx - a^3y^4 - a^2y$.

119. $ax - by - cx - dy$.

Write each of the following expressions as the difference of two squares:

120. $x^2 - a^2 - 2ab - b^2$.

121. $a^4 - 2a^2b^2 + b^4 - c^4$.

122. $x^2 + 9 - 6x - a^2$.

123. $4x^2 + 1 - b^2 - 4x$.

124. $25 + 2cd - c^2 - d^2$.

125. $x^2 + 4y^2 - z^2 - 4xy$.

2. FRACTIONS

1. State and illustrate the Fundamental Principle of Fractions.

2. Indicate which of the following are *true* and which are *false* by writing *T* for "True" and *F* for "False":

(1) $\frac{a+2}{2} = a.$

(2) $a - \frac{1}{2}a = \frac{1}{2}a.$

(3) $\frac{a^2+b^2}{a+b} = a+b.$

(4) $\frac{a}{b} = \frac{a^2}{b^2}.$

(5) $\frac{a}{a-b} = \frac{-a}{b-a}.$

(6) $\frac{a}{b} + \frac{a}{c} = \frac{ac+ab}{b+c}.$

(7) $\frac{a+b}{c} \times \frac{c}{a+b} = 0.$

(8) $\frac{cx+y}{c} = x+y.$

(9) $\frac{m-n}{c} = \frac{m}{c} - \frac{n}{c}.$

(10) $\frac{1}{c-d} + \frac{2}{d-c} = \frac{3}{c-d}.$

Reduce each of the following fractions to its lowest terms:

3. $\frac{8xyz}{xy^2}.$

4. $\frac{l+15lw}{3l^2}.$

5. $\frac{3x}{x^2-3x}.$

6. $\frac{w(x+y)}{x+y}.$

7. $\frac{x^2-6x}{x^2-7x+6}.$

8. $\frac{ax^2-ay^2}{4a(x-y)^2}.$

9. $\frac{2x^2+5x+2}{3x^2+12x+12}.$

10. $\frac{3m^2-7mr-6r^2}{m-3r}.$

11. Express with a positive denominator $\frac{2b-a}{-3}.$

12. Change the fraction $\frac{3}{x+9}$ to one having $2x^2+13x-45$ for its denominator.

In each case combine the fractions as indicated:

13. $\frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3}.$

14. $\frac{3}{4b} + \frac{2}{3b^2} - \frac{5}{6b^3}.$

15. $7 - \frac{x}{y} + \frac{1}{y^2}.$

16. $x - \frac{x+y}{x} - 6.$

17. $\frac{4x+3}{5} - \frac{2x-3}{6}.$

18. $\frac{3}{a-b} + \frac{2}{a+b}.$

19. $\dfrac{2}{3a-7}-\dfrac{2}{3a+2}$.

20. $\dfrac{1-a}{3-6a}+\dfrac{1+a}{2a-1}$.

21. $\dfrac{2}{(a-1)^2}+\dfrac{3}{a^2-3a+2}$.

22. $\dfrac{x+y}{x-y}+\dfrac{4xy}{y^2-x^2}-\dfrac{x-y}{x+y}$.

23. $\dfrac{2a-1}{a-2}-\dfrac{2a-5}{a-4}$.

24. $\dfrac{ad-bc}{2cd(c+d)}+\dfrac{ad+bc}{2cd(c-d)}$.

25. Combine $\dfrac{x+2}{3}-\dfrac{3x+5}{4}-\dfrac{5-2x}{6}+\dfrac{7-2x}{8}-\dfrac{3-7x}{12}$.

26. If x is a positive fraction less than 1, how does x^2 compare in value with x?

Perform the indicated operations in each case:

27. $\dfrac{2x^2}{3y^2}\cdot 6xy^2$.

28. $\dfrac{x^2+2x+1}{x^2-1}\cdot\dfrac{5x}{x+1}$.

29. $\dfrac{a^2-9}{2a+2}\div\dfrac{a+3}{2}$.

30. $\dfrac{7st^2}{10hk}\div\dfrac{14t^3}{5h^2k^3}$.

31. $\dfrac{3x^2}{7c}\div\dfrac{2x}{21c^2}$.

32. $\dfrac{3(x+y)}{4}\cdot\dfrac{4}{5(x+y)}$.

33. $\dfrac{a^2+2a-3}{b}\div\dfrac{4a+12}{2b^2}$.

34. $\dfrac{M^2-4}{2M^2+6}\div\dfrac{M+2}{M^2+3}$.

35. $\dfrac{56}{r+3}\div\dfrac{8}{9-r^2}$.

36. $\dfrac{21n^2}{3n+21}\div\dfrac{7n^2-49n}{n^2-49}$.

37. $\dfrac{4a^2-9}{9a^2-4}\times\dfrac{2a-3a^2}{9-9a+2a^2}\div\dfrac{a^2}{3a-a^2}$.

38. Divide $\left(1+\dfrac{1}{x-1}\right)$ by $\dfrac{2}{x-1}$, and express the resulting fraction in its lowest terms.

39. Reduce to simplest form: $\left(1-\dfrac{t}{s+t}\right)\left(\dfrac{s+t}{s}\right)$.

Simplify each of the following expressions and express the result as a fraction in its lowest terms:

40. $\left(\dfrac{5}{x^2-4}+\dfrac{2}{2-x}\right)\div\left(2+\dfrac{3}{x-2}\right)$.

41. $\left(\dfrac{1}{2a-b}-\dfrac{5}{6a}\right)\div\left(\dfrac{5}{2a^2}-\dfrac{3}{2a^2-ab}\right)$.

42. $\left(x - \frac{1}{x}\right)\left(x - \frac{x}{x+1}\right)$.

43. $\left(4 - \frac{6}{c+1}\right) \div \left(8 - \frac{4c-8}{c^2-1}\right)$.

Simplify each of the following:

44. $\dfrac{\frac{1}{5} + \frac{1}{2a}}{4a - \frac{25}{a}}$.
45. $\dfrac{\frac{18c}{5} - 2}{\frac{3c}{5} + 3}$.
46. $\dfrac{\frac{2}{a+2} + 2}{\frac{6}{a+2}}$.

47. Indicate which of the following fractions can be simplified by canceling the 5's:

(1) $\frac{5xy}{5ab}$. (2) $\frac{5x}{x-5}$. (3) $\frac{5n}{5(a+b)}$. (4) $\frac{a-5}{b-5}$.

48. Indicate which of the following fractions can be simplified by canceling the n's:

(1) $\frac{a+n}{b+n}$. (2) $\frac{an}{bn}$. (3) $\frac{n-5}{n^2+6}$. (4) $\frac{n(a+3)}{4n(c-5)}$.

3. EXPONENTS AND RADICALS

Find the square roots of the following:

1. 2601.
2. 68.89.
3. 8281.
4. 3.2041.
5. 69,696.
6. 14.8225.
7. 164,025.
8. 224,676.

Find to the nearest hundredth the square roots of the following:

9. 145.
10. 156.
11. 59.21.
12. 250.
13. 68.59.
14. 359.
15. 1749.
16. 25,267.

Perform the indicated operations:

17. $a^5 \cdot a^2$.
18. $3a^4 \cdot a$.
19. $4x \cdot 5x^3b$.
20. $8y^3 \cdot (-3y^4)$.
21. $x^6 \div x^2$.
22. $(-20a^5) \div (-5a^5)$.
23. $(-4a)^3$.
24. $-(5a)^2$.
25. $4^2 \cdot (5a)^2$.
26. $m^3n \div m^2$.
27. $3x^n \cdot 2x^n$.
28. $16y^{2n} \cdot (-3y^{n+1})$.

Simplify the following:

29. $\sqrt{40}$.
30. $\sqrt{108}$.
31. $2\sqrt{27}$.
32. $4\sqrt[3]{-16}$.
33. $-10\sqrt{a^4b^5}$.
34. $\sqrt[3]{81\,ab^4}$.
35. $\frac{1}{4}\sqrt{56\,x^3}$.
36. $-a\sqrt[3]{-a^3b^4}$.
37. $c^3\sqrt{c^7d^9}$.
38. $2\sqrt{\frac{1}{8}}$.
39. $14\sqrt{\frac{2}{7}}$.
40. $\dfrac{5\sqrt{7}}{\sqrt{6}}$.
41. $12\sqrt{\frac{5}{18}}$.
42. $6\sqrt[3]{\frac{2}{9}}$.
43. $\dfrac{\sqrt{12}}{\sqrt{5}}$.
44. $8\sqrt{\dfrac{a^2}{2}}$.
45. $5\sqrt{\dfrac{y^3}{5}}$.
46. $4\sqrt[3]{\dfrac{x}{2}}$.

In the following expressions combine the terms as indicated:

47. $12\sqrt{2} - 2\,d\sqrt{2} + 18\sqrt{2} - 5\,d\sqrt{2}$.
48. $7\,c\sqrt{3} - 5\,c\sqrt{3} + (3\,c - 2)\sqrt{3}$.
49. $7\sqrt{3} + 2\,a\sqrt{3} - (3 - a)\sqrt{3}$.
50. $\sqrt{24} - 3\sqrt{6} + 4\sqrt{54}$.
51. $12\,x\sqrt{2} + 16\,x\sqrt{\frac{1}{2}} - 7\sqrt{18\,x^2} - 12\,x\sqrt{\frac{9}{2}}$.
52. $\sqrt{25\,a^2} + \frac{1}{4}\sqrt{98\,a^2} - 3\sqrt{a^2 - 4\,a + 4}$.
53. $6\sqrt{\frac{3}{2}} - \sqrt{24} + 3\sqrt{\frac{2}{3}} + 5\sqrt{6}$.
54. $3\sqrt{5} + 2\sqrt{3} + 7\sqrt{5} - \sqrt{3}$.

Perform the indicated operations and simplify results:

55. $2\sqrt{6}\cdot 5\sqrt{2}$.
56. $\frac{1}{2}\sqrt{2}\cdot\sqrt{3}$.
57. $5\sqrt{3\,x}\cdot\sqrt{3\,x^5}$.
58. $9\sqrt{cd^2}\cdot\sqrt{c^3}$.
59. $\sqrt{54} \div \sqrt{3}$.
60. $\sqrt{24\,x^3} \div \sqrt{2\,x}$.
61. $10\sqrt{a^7} \div 4\sqrt{a}$.
62. $12\sqrt{48\,a^4} \div 9\sqrt{3\,a^3}$.
63. $\sqrt[3]{4\,a^5}\cdot\sqrt[3]{2\,a^4}$.
64. $9\,x^3y^2z\sqrt[3]{x^6} \div \sqrt{y^4z^2}$.
65. If $x = 4$, find the value of

$$2\sqrt{\frac{9}{x}} - \sqrt[3]{\frac{-2}{x^2}}.$$

Perform the indicated operations and simplify results:

66. $(\sqrt{a}+b)(\sqrt{a}-b)$.
69. $(2\sqrt{5}+2)(3\sqrt{5}-3)$.
67. $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})$.
70. $(4\sqrt{6}+3\sqrt{5})(4\sqrt{6}-3\sqrt{5})$.
68. $(\sqrt{5}+3)^2$.
71. $(3\sqrt{2}-2\sqrt{6})(3\sqrt{2}-\sqrt{6})$.

Simplify each of the following:

72. $a^{\frac{1}{2}}b^{\frac{1}{3}} \cdot a^{\frac{2}{3}}b^{\frac{2}{3}}$.
74. $(4\,m^2n^{\frac{1}{3}}r) \cdot (-2\,mn^{\frac{2}{3}}r^2)$.
73. $x^{\frac{2}{3}}y^{\frac{3}{4}} \cdot x^{\frac{1}{3}} \cdot y^{\frac{1}{4}}$.
75. $(3\,a^2b^{\frac{2}{3}}c^{n+3})(-4\,a^5b^{\frac{1}{3}}c^{n-2})$.

76. Find the numerical value of $4^{\frac{1}{2}}+8^{\frac{1}{3}}+16^{\frac{1}{2}}-3(9^{\frac{1}{2}})+100^{\frac{1}{2}}$.

77. Simplify $\sqrt{4\,x}+4\,x^{\frac{1}{2}}-4\sqrt{x}+(2\,x^{\frac{1}{4}})^2$.

III. Equations

In each of the following exercises select from the values of x in the parentheses the one that satisfies the given equation:

1. $3\,x+8=7$. $x=(+\frac{1}{3}, -\frac{1}{3}, +5, -5)$.
2. $13\,x-6=5\,x-6$. $x=(0, 1\frac{1}{2}, 1\frac{2}{3}, -3)$.
3. $3=\frac{10}{x}-10$. $x=(2, \frac{1}{2}, \frac{10}{13}, 1\frac{3}{10})$.
4. $\frac{1.8}{x}-.4=5$. $x=(+\frac{1}{3}, -\frac{1}{3}, +3, -3)$.
5. $\frac{3}{5}\,x=1-\frac{2}{5}\,x$. $x=(-\frac{1}{5}, +\frac{1}{5}, -1, +1)$.

Solve each of the following equations, and check each solution.

6. $5\,a-4=a$.
12. $n+10+5\,n=37-3\,n$.
7. $7\,x=18+x$.
13. $4\,p-12+7\,p=p+13$.
8. $5\,c+3=2\,c-9$.
14. $8\,x+11-5\,x-2=45$.
9. $4\,x-2=7-3\,x$.
15. $.3\,n-4=.5\,n-8$.
10. $10\,a=20-5\,a$.
16. $.05\,x-13=.02\,x+2$.
11. $5\,x-3+2\,x=x+7$.
17. $7\,x-4+5\,x=9\,x-12+x$.
18. $5\,t+9-2\,t=44+10\,t+21$.
19. $11\,n+19=-7\,n+10+9\,n$.

20. $4x + x(x - 3) = 7 + (3 + x^2)$.

21. $6y^2 - 3y(2y + 1) = 5(8 - y)$.

22. $5(a^2 - 4) - a(2 + 5a) + 8 - a = 0$.

23. $(x + 9)(x + 2) - (x - 3)(x + 7) = 11$.

24. $3(x - 3) - 2(x - 2) = 7(x - 2) - 4(x - 2)$.

25. $(w + 2)^2 - (w + 3)^2 = 13$.

26. $(x + 1)^2 + 54 = (x + 2)(5x - 3) - 4(x - 1)^2$.

27. $(4x - 9)(x + 3) - 3(x - 4) = (2x - 5)^2$.

28. What must be the value of c in the equation $3x - c = 7$ if $3\frac{1}{3}$ is the root of the equation?

29. What is the value of n in the equation $4x - 5 = nx - 3$ when $x = 2$?

30. When $x = 6$, what must be the value of a in the equation $x(x - 2) = (x - 3)(x - 2) + ax^2$?

Express each of the following equations in words:

31. $10 + n = 12$.

32. $5n - 1 = 9$.

33. $7 - \frac{x}{2} = 6$.

34. $x + 2(x - 7) = 20$.

35. State and illustrate the Rule of Equality which is used in solving equations.

36. What is meant by "clearing an equation of fractions"?

37. What is meant by "checking the solution" of an equation?

Solve each of the following equations, and check each result:

38. $\frac{5n}{2} = 22 + \frac{7n}{5}$.

39. $\frac{7x}{12} - \frac{1}{4} = 2x - \frac{5}{3}$.

40. $\frac{3x}{2} - \frac{2x}{3} = 6$.

41. $a + \frac{a - 3}{4} = 3$.

42. $\frac{5v}{3} - 6 = \frac{7v}{9} + 2$.

43. $\frac{4x}{7} + \frac{11}{5} = \frac{6x}{5}$.

44. $\frac{2x + 4}{4} = \frac{x}{5} + 4$.

45. $\frac{2x}{3} - \frac{3x - 5}{2} = 0$.

46. $\frac{7x}{10} - \frac{2x - 1}{6} = 1 + \frac{x + 7}{12}$.

47. $\frac{2s - 1}{2} - \frac{s - 7}{8} = 6\frac{1}{2}$.

48. $\frac{5(3x+2)}{4} - \frac{4(5x-4)}{3} = \frac{2(4x+1)}{9}$.

49. $\frac{x-2}{2x} - \frac{x+3}{3x} = \frac{5}{6}$.

50. $\frac{1}{3}(x+1) - \frac{1}{6}(x+1) = 6$.

51. $\frac{4x-7}{4} - \frac{x+7}{8} = 0$.

52. $.6x - 1.5 = .3 - .15x$.

53. $\frac{16x+3}{10} - \frac{2x-5}{5x-1} = \frac{8x-1}{5}$.

54. $\frac{6x+2}{x-3} + \frac{8}{x+3} = 17$.

55. $\frac{8}{x-2} - \frac{1}{x-5} = \frac{7}{x+1}$.

56. $\frac{1}{x+1} - \frac{1}{1-x} = \frac{-8}{x^2-1}$.

Find the value of x in each of the following equations:

57. $4 : 9 = 3 : x$.

58. $5 : 7 = x : 3$.

59. $18 : x = 3 : 10$.

60. $a : c = x : b$.

61. $m : p^2 = x : q$.

62. $a^2 : b^2 = n : x^2$.

63. $r : s = x^2 : 4$.

64. $(x+1) : 5 = (x-3) : 2$.

Solve the following literal equations for x or y:

65. $7y - 3a^2 = 11a^2$.

66. $5x - 9b = 5c$.

67. $a + 5y = 3y + b$.

68. $\frac{ax}{2b} - 4b^2 = \frac{2bx}{a} - a^2$.

69. $8r - 3(y + 2r) = 5r$.

70. $4(6x - 3a) = 3(5x - a)$.

71. $ax + b^2 = a^2 - bx$.

72. $\frac{c-x}{nx} - \frac{n}{cx} = -\frac{1}{c}$.

Solve the following pairs of equations, and check each set of results:

73. $2x + 7y = 25,$
$2x + y = 13.$

74. $5x = y,$
$y = 12 - x.$

75. $y = 8 - 2x,$
$y = 6 - 3x.$

76. $3x = 5 + 2y,$
$2x - 3y = 0.$

77. $2x + 5y = 1,$
$6x + 7y = 3.$

78. $4x - y = 9,$
$2x - 3y = -23.$

79. $4x + 3y = 21,$
$9y - 2x = 14.$

80. $0.3x + 0.2y = 9.5,$
$0.2x + 0.3y = 10.5.$

81. $2x - y = c,$
$x + 3y = d.$

82. $Lx + y = -D,$
$x + Ly = D.$

83. $\frac{x-3}{5y} = -2,$
$x + 7y = 6.$

84. $\frac{x+y}{3} - \frac{x-y}{4} = 11,$
$\frac{x+y}{2} - \frac{x-y}{3} = 8.$

85. $\frac{2x+y}{x-y} = 3,$
$\frac{3x-2y}{y+1} = 5.$

86. $by - ax = 2b,$
$\frac{x}{b} + \frac{y}{a} = \frac{2}{b}.$

Solve each of the following equations:

87. $x^2 + 11 = 60.$

88. $2x^2 + 7 = 15.$

89. $0.04 + y^2 = 0.85.$

90. $y^2 - 1.21 = 0.$

91. $x^2 + \frac{1}{2} = \frac{3}{4}.$

92. $2y^2 - 5 = 7.$

Solve by factoring:

93. $x^2 - 7x = 0.$

94. $4a^2 - 20a = 0.$

95. $2x^2 + x - 21 = 0.$

96. $x^2 - 6x - 6 = 10.$

97. $6x^2 + 5x - 4 = 0.$

98. $x^2 + 2a^2 = 3ax.$

99. $x^2 + 2bx = 15b^2.$

100. $10x^2 + ax = 3a^2.$

101. State the quadratic formula.

In each of the following exercises supply in the parentheses the expression which is necessary to produce a trinomial square, and give the square root of the resulting trinomial:

102. $x^2 + 12x + (\quad).$

103. $x^2 - (\quad) + 4.$

104. $y^2 - 5y + (\quad).$

105. $4x^2 + 4x + (\quad).$

106. $.01x^2 - (\quad) + 81.$

107. $\frac{1}{4}r^2 + r + (\quad).$

Solve by completing the square or by the quadratic formula:

108. $x^2 - 10x + 2 = 0$.
109. $x^2 + 3x - 5 = 0$.
110. $x^2 + 4x = 90$.
111. $3x^2 - 5x - 4 = 0$.
112. $2x^2 + 3x = 7$.
113. $x^2 - 4x = 8$.
114. $5x^2 - 8x + 1 = 0$.
115. $6x^2 - x - 2 = 0$.
116. $2x^2 + 6x - 3 = 0$.
117. $x(14 - x) = 47$.
118. $2x^2 - 5bx = 12b^2$.
119. $x^2 - 2ax = b^2 - a^2$.
120. $ax^2 - bx - c = 0$.
121. $4x : 5(1 + 2x) = 1 : 6(x - 2)$.

IV. The Formula

1. Write a formula for the perimeter (p) of a rectangle whose dimensions are w and $w + 5$.

2. Write a formula for the area of a square whose side is $s + 2$.

3. Write a formula for the area of a circle whose radius is $a + b$.

4. The dimensions of a rectangular solid are w, $w + 3$, $2w - 5$. Write a formula for (1) its total surface (S); (2) its volume (V).

5. An automobile was worth d dollars when bought. If its value decreases $200 each year, write a formula for its value (V) after y years.

6. Write a formula for the cost (c) of sending a telegram of w words if 30¢ is charged for the first ten words and 3¢ for each additional word.

7. The rule for finding the horse power of a certain type of gas engine is as follows: Multiply the square of the diameter (in inches) by the number of cylinders and divide by 2.5. State this rule as a formula, letting P represent the horse power, d the diameter in inches, and n the number of cylinders.

8. Write the formula for the number of revolutions made by a wheel in going a certain distance, using n for the number of revolutions, d for distance, and r for the radius of the wheel.

9. Write the formula which shows the number of cents (c) in d dollars, q quarters, and n nickels.

10. Express the following rule as a formula: The dividend (D) equals the product of the divisor (d) and the quotient (Q), plus the remainder (R).

11. To find the approximate number of bushels (b) in a bin, multiply the length (l) by the width (w) by the height (h), each expressed in feet, and divide this product by 1.25. Write this rule in algebraic shorthand.

12. In the formula $i = prt$, if $p = \$500$, $r = .06$, and $t = 2$, find i.

13. In the formula $A = \frac{bh}{2}$, $A = 8$ and $h = 3$; find b.

14. By means of the formula $A = \pi r^2$, find the area of a circle whose radius is 5.2.

15. If $A = \frac{1}{2}(b + b')h$, find the value of A when $b = 41$, $b' = 13$, and $h = 12$.

16. When $V = 4$ and $t = 1$, find the value of N, given that $N = (V + t)(V - t)$.

17. Using the formula for the volume of a cylinder, $V = \pi r^2 h$, find the value of V if $\pi = 3.14$, $r = 4$, and $h = 8$.

18. Given the formula $C = \frac{5}{9}(F - 32)$. Find F when C is 20.

19. In the formula $l = a + (n - 1)d$, what is the value of a if $l = 24$, $n = 9$, and $d = 2$?

20. The volume of a metal plate is given by the formula $V = wt(h + l - t)$. How many cubic inches of metal are in the plate if $l = 8$ inches, $h = 6$ inches, $w = 4$ inches, and $t = 1\frac{1}{2}$ inches?

21. The formula for the volume of a cylinder is $V = \pi r^2 h$. If a cylinder contains 3297 cubic feet and is 15 feet high, find the radius of the base to the *nearest tenth* of a foot.

22. In the formula $I = \frac{E}{R + nr}$, if $E = 220$, $R = 40$, $n = 30$, and $r = 2$, find I.

23. Given the formula $W_1 d_1 = W_2 d_2$. Find W_1 if $W_2 = 80$, $d_2 = 3$, and $d_1 = 4$.

24. Given the formula $S = \frac{n(a + l)}{2}$. Find S if $n = 20$, $a = 4.18$, and $l = 92$.

25. Using the formula $v = \frac{s + 16\,t^2}{t}$, find v when $s = 95$ and $t = 1\frac{1}{4}$.

26. Study the relation between the corresponding values of d and l in the following table, and then express this relation by means of a formula:

If $l =$	6	10	16	20	32
then $d =$	6	8	11	13	19

27. A tank contains 50 gallons of water. It is being filled at the rate suggested by the following table:

If $m =$	0	1	2	3	4	6	
then $g =$	50	65	80	95	110		200

(1) What is the formula for the number of gallons (g) which are in the tank after any number of minutes (m)?

(2) Find the missing numbers in the table.

28. Write the formula suggested by the following table, and then find the missing values:

When $x =$	6	9	15	21	27	30	
then $y =$	0	2	6	10			36

29. Express as a rule the formula $V = \frac{4}{3}\,\pi r^3$, if V represents the volume of a sphere and r represents the radius.

30. The total area of the walls and the ceiling of a room which is l feet long, w feet wide, and h feet high is found by the formula $A = lw + 2\,hl + 2\,hw$. Express this formula as a rule.

31. Solve for p the formula $r = \frac{v^2 pL}{a}$.

32. Solve the formula $\frac{E}{e} = \frac{R + r}{r}$ for r.

33. Solve the formula $A = P + Prt$ for P.

34. Solve the formula $\frac{1}{f} + \frac{1}{p} = \frac{1}{q}$ for p.

35. Solve for m the formula $u = v\left(\frac{m}{M + m}\right)$.

36. Solve $h = \frac{wv^2}{1.25\,r}$ for v.

37. Solve $L = \frac{Wh}{d(W + P)}$ for P; for h.

38. Solve $w = \frac{.08\,d^2}{l}$ for l; for d.

39. Solve $M = \frac{D^3R}{125}$ for R.

40. Solve $W = \frac{V^2}{R}$ for V.

41. Solve $Q = .327\,vd^2$ for v; for d.

42. Solve the formula $S = 2\,\pi r^2 + 2\,\pi rh$ for r.

43. If the sides of a triangle are represented by a, b, and c, and $s = \frac{1}{2}(a + b + c)$, the formula for the area of the triangle is $A = \sqrt{s(s - a)(s - b)(s - c)}$. Find A if (1) $a = 13$, $b = 30$, and $c = 37$; (2) $a = 15$, $b = 41$, and $c = 52$.

44. Given the formula $V = 1.41\sqrt{\frac{E}{D}}$. Solve for D.

45. Given the formula $d = rt$.

(1) Upon what does the value of d depend?

(2) What is the effect upon d if t remains constant and r is doubled? r is halved? r is replaced by $5\,r$?

(3) What is the effect upon d if r and t are both doubled?

(4) How is d affected if r is doubled and t is halved?

46. Given the formula $h = \frac{V}{lw}$.

(1) What is the effect upon h when V is doubled and l and w remain the same?

(2) How is h affected when V and l remain constant but w is increased?

47. Given that $\tan A = \frac{a}{b}$. How is the value of the tangent affected when a is increased and b remains constant?

48. Given the formula $Q = .61\ A\sqrt{2\ gh}$. (1) Upon what does the value of Q depend? (2) Will doubling A double Q?

49. In the formula $s = \frac{n}{2}(a + l)$ express n in terms of s, a, and l.

50. Solve for W the formula $R = \frac{WL - X}{L}$.

51. Solve for h the formula $k(h - 1) = R$.

52. Solve for b the formula $S = \frac{a(b + c)}{2}$.

53. Solve for M the formula $V = \frac{h}{6}(B + b + 4\ M)$.

54. Solve for h the formula $V = \frac{\pi r^2 h}{3}$.

55. In the formula $p = \frac{qs^2}{t^2}$ find t when $p = 5$, $s = 8$, and $q = 11\frac{1}{4}$.

56. In the formula $A = \pi r^2$, if r is doubled, how does A change?

57. If y and x are positive integers and $y = \frac{2}{3\ x + 1}$, does y become larger or smaller as x becomes larger?

58. If y and x are positive integers and $y = \frac{1}{x}$, does y become larger or smaller as x becomes larger?

59. If $y = -12\ x + 50$ and x is a positive integer, does y increase or decrease as x increases?

60. If a is greater than b, and if $x = (a - b)^2$ and $y = (b - a)^2$, how does x compare in value with y?

61. (1) Given the formula $s = \frac{1}{2}\ vt$. Find t in terms of the other letters. Substitute this value of t in the formula $s = \frac{1}{2}\ at^2$, and thus derive a formula for a in terms of v and s.
(2) Find a if $v = 24$ and $s = 8$.

62. If $F = \frac{2 + \frac{1}{a}}{3}$, how does F change as a increases?

V. The Graph

1. The number of passenger automobiles, in thousands, produced in the United States in ten successive years, beginning with 1917, was as follows: 1741, 926, 1658, 1883, 1444, 2406, 3486, 3243, 3839, 3937. Represent these statistics graphically.

2. The average price received by the American farmer for 1 bushel of wheat, in ten successive years, beginning with 1918, was as follows: \$2.01, \$2.06, \$2.33, \$1.49, \$0.95, \$1.04, \$0.97, \$1.62, \$1.58, \$1.22. Represent these statistics graphically.

3. The number of pupils enrolled in the high schools of one city during eight successive years, beginning with 1916, was as follows: 1147, 1403, 1470, 1644, 2627, 2656, 4342, 5522. Draw a graph based on these figures.

4. The remarkable growth in attendance in four-year public high schools in the United States is shown by the following table:

Year	1916	1918	1920	1922	1924	1926
Boys	618,851	671,774	791,343	1,004,355	1,156,300	1,418,427
Girls	743,663	892,378	989,989	1,156,506	1,319,086	1,581,567
Total	1,362,514	1,564,152	1,781,332	2,160,861	2,475,386	2,999,994

Using the same axes, draw a set of graphs picturing the statistics given in this table. (Let one division of the vertical axis represent 100,000 pupils.)

Each of the following tables is based on two related sets of numbers. Draw a graph of each of these tables. By means of the graph supply the missing numbers. In each case let the upper row of numbers be pictured on the horizontal axis, while the numbers of the lower row are pictured on the vertical axis.

5.

If t is	0	1	2	3	4	5	6
then d is	5	7	9	11	13		

6.

If w is	0	1	3	5	10	12	
then p is	-5	-3	1	5	15		41

7.

When d is . . .	0	2	4	6	8	12	
then e is	2	3	4	5	6		44

8.

When x is . . .	0	1	2	3	4	5	6
then y is	0	1	4	9	16	25	36

9.

When x is . . .	-3	-2	-1	0	1	2	3
then y is	12	7	4	3	4	7	12

10.

When x is . . .	0	1	2	3	4	5	6
then y is	1	2	4	8	16	32	64

11. A pupil's average standing in algebra during each of the first six weeks of the school year is given in the following table:

1st week	86	4th week	88
2d week	75	5th week	90
3d week	80	6th week	92

Represent this information by means of a bar graph.

12. Is the graph of the formula $d = 5r$ a broken line, a straight line, or a curved line?

13. What is the value of x at the point where the graph of $y = 2x + 4$ crosses the x-axis?

14. Given the formula $c = 3n - 5$.

(1) Make out a table of values for the formula for $n = 0, 5, 10, 15, 20, 25$.

(2) Draw the graph of the formula, showing values of c on the vertical axis and values of n on the horizontal axis.

(3) From the graph determine the value of c when $n = 18$.

(4) From the graph determine the value of n when $c = 34$.

15. If a man swims at the rate of 2 miles an hour, the formula $d = 2\,t$ represents the relation between the distance (d) and the time (t).

(1) Construct a table that will give the values of d corresponding to $t = 1, 2, 3, 4, 5, 6$.

(2) Plot the graph of this table.

(3) Find from the graph the time needed to cover 7 miles.

16. The cost of setting up the type for a certain pamphlet and of printing copies is given by the formula $c = .10\,n + 10$, where n represents the number of copies and c the total cost in dollars.

(1) Construct a table of values for the formula, choosing values of n from 0 to 40 at intervals of 10.

(2) Plot the graph.

(3) From the graph find (a) the cost of printing 25 copies; (b) the cost of setting up the type.

17. The formula $A = s^2$ expresses the area (A) of a square in terms of a side (s).

(1) Plot the graph of this formula from $s = 0$ to $s = 9$ inclusive.

(2) From the graph estimate the area (A) when $s = 2\frac{1}{2}$.

(3) From the graph estimate the side (s) when $A = 12$.

18. Plot the graph of the equation $y = 5\,x - 4$ for values of x from -3 to 5 inclusive.

From the graph find the value of x for which $y = 3$.

19. Plot the graph of the equation $y = x^2 - 4\,x - 5$. By means of this graph find the roots of the equation $x^2 - 4\,x - 5 = 0$.

20. Solve graphically the following pairs of equations:

(1) $x + y = 5\frac{1}{2}$, $2\,x - y = 2$.

(2) $x + 2\,y = 5$, $3\,x - 4\,y = 11$.

21. Solve graphically the following pairs of equations:

(1) $x - 2\,y = 7$, $4\,x + y = 1$.

(2) $2\,x + y = 3$, $x - 3\,y = 8\frac{1}{2}$.

22. Plot the graph of the equation $y = 2\,x^2 - 12\,x + 20$ for values of x from 0 to 6 inclusive, and estimate from your graph the values of x for which $y = 6\frac{1}{2}$.

23. If in the formula $s = vt - \frac{1}{2}at^2$, it is known that $v = 6$ and $a = 2$, plot the graph of the resulting equation for values of t from 0 to 6.

24. Plot the graph of $y = x^3 - 6x^2 + 9x + 1$, using for x the values 0, 1, 2, 3, 4. From the graph estimate the values of x when $y = 2$.

VI. Problems

1. A debt of $1500 is to be paid in five annual installments. Each year the payment is to be increased by $50. How many dollars must be paid each year?

2. A family budget allows twice as much for food as for housing, and three times as much for all other items as for housing. If the total budget is $3600, how much money should be allowed for each item?

3. In a class of 44 pupils the number of boys is 16 less than twice the number of girls. How many boys and how many girls are in the class?

4. The total cotton crop of the world for a certain year was 16 million bales. The United States produced three times as much cotton as the rest of the world. How many million bales did the United States produce?

5. The cost of sending a parcel to the third zone, expressed in cents, equals 6 added to twice the number of pounds that the parcel weighs. What is the weight of a parcel costing 36¢?

6. The sum of two numbers is 12, and their difference is 4. Find the larger number.

7. Find the number which, if divided by 7, gives a quotient of 9 and a remainder of 3.

8. The difference between two positive numbers is 2, and the difference between their squares is 8. Find the numbers.

9. Find a fraction whose value is $\frac{9}{16}$ and whose denominator is 32 less than twice its numerator.

10. Francis is able to save 50 cents a week. He has $2 in the school bank now. How many weeks will it take before he will have a total deposit of $20?

11. We wish to drive to a town 120 miles away from our home. If we are to leave at 1 P.M., at what rate must we travel to reach our destination by 5 P.M.?

12. A box is 2 feet long and $1\frac{1}{2}$ feet wide. How high must it be to hold 540 pounds of sugar? (One pound of sugar occupies 20 cubic inches.)

SUGGESTION. Express all dimensions in inches.

13. (An old puzzle.) A cork and a bottle together cost $1.10. The bottle cost $1 more than the cork. Find the price of the cork and of the bottle.

14. A book and a pencil together cost 60 cents. The book cost 50 cents more than the pencil. Find the price of the book and of the pencil.

15. William Brown cleaned his father's automobile. After William had been working for a while, his brother Dick joined him and they completed the work together. When the job was finished, the boys received from their father $1.75. If Dick received 35 cents less than William, how much did each boy earn?

16. Cotton loses one third of its weight when ginned and baled. How many pounds of cotton must be picked to yield 6 bales? (One bale contains 500 pounds.)

17. A troup of Boy Scouts walked from their camping ground to a railroad station, a distance of 5 miles. There they took a train. At the end of 8 hours they arrived at their home town, 245 miles from camp. At what rate did the train travel?

18. A rectangle is $1\frac{1}{5}$ times as long as it is wide. If 2 feet are taken from each dimension, the new rectangle thus formed will have $\frac{2}{3}$ of the area of the original one. Find the area of each.

19. A bank will loan, for the purpose of building, an amount of money equal to one third of the value of the property. A man has a lot worth $2000, and has saved an additional sum of $8000 for building a home on this lot. If he depends entirely upon a bank loan to supply the remaining funds, how expensive a house can he build?

20. A and B, partners in a business, agree to divide their profits so that A will have 30 % more than B, since A owns the building. How should they divide a profit of $18,400?

21. A man invests $\frac{4}{5}$ of his capital at $4\frac{1}{2}$ % and the rest at 4 %. His annual income is $176. Find his capital.

22. A man invests $\frac{2}{5}$ of his property at 4 %, $\frac{1}{3}$ at 5 %, and the remainder at 3 %. If his resulting annual income is $610, find the value of his property.

23. In how many years will a sum of money double itself at 6 % simple interest?

24. A man invested $10,000 in two enterprises. The first pays him 6 % on his investment and the second 5 %. If his annual income from both investments is $560, how much did he invest in each?

25. Mr. Hill invested $6500 in stocks and bonds. The stocks paid 8 % and the bonds 5 % on the money invested. If the yearly income from the bonds was $65 more than the income from the stocks, how much money was invested in each?

26. The sums of $400 and $600 are invested at different rates and bring an annual income of $44. If the investments were interchanged, the annual income would be increased by $2. Find the rates.

27. The sum of the present ages of Frank and his father is 58. In 10 years Frank will be $\frac{1}{2}$ as old as his father. Find their present ages.

28. One pound of coffee and 5 pounds of sugar together cost 70 ¢. After the price of sugar has advanced $12\frac{1}{2}$ % and the price of coffee 20 %, 3 pounds of sugar and 2 pounds of coffee together cost 99 ¢. Find the price of each per pound.

29. I changed a 5-dollar bill and received in return 35 coins consisting of quarters and dimes. How many of each kind did I receive?

30. During their vacation Harold and James work in a grocery store. Harold receives 10¢ an hour more than James. In 3 days of 8 working hours a day, they together earn $16.80. How much does each earn an hour?

31. Two motor boats start at the same time and place and travel in opposite directions. The ratio of their rates is 2 : 3. In 5 hours they are 100 miles apart. Find the rate of each.

32. Two trains start at the same time from stations that are 288 miles apart and travel toward each other. They meet in 4 hours from the time they started. If the rate of one is 6 miles an hour less than the rate of the other, what is the rate of each train?

33. A man walks at the rate of 4 miles an hour. How far can he walk into the country, if he can ride back on a trolley that travels at the rate of 20 miles an hour, provided he wishes to return to his home 3 hours from the time he started?

34. A travels at the rate of x miles an hour, B at the rate of y miles an hour. In 5 hours A can travel as far as B can travel in 7 hours. If A travels 4 miles an hour faster than B, find x and y.

35. A can do a piece of work in 6 days and B can do it in 10 days. How long will it take them to complete the job when both work together?

36. A printing press did a certain piece of work in 4 hours. When a second press was installed, a similar job was completed by both machines in $2\frac{1}{2}$ hours. In how many hours could the second press have done the work alone?

37. A and B can do a piece of work in 4 days. A works alone for 3 days and is then joined by B. The two men complete the work in 3 days more. In what time could each do the work alone?

38. How many pounds of 45-cent coffee and how many pounds of 25-cent coffee must be mixed together to make 50 pounds worth 33¢ a pound?

39. A baking company wished to make up 1500 one-pound boxes of assorted cakes to sell for 29¢ a box. Cakes were to be used which retailed at 35¢ and 25¢ a pound. How many pounds of each kind should be used?

40. How much water must be added to 25 pounds of an 8% solution of salt to make a 5% solution?

41. Mary and Jane are playing seesaw. Mary, who knows that her weight is 78 pounds, is $5\frac{1}{2}$ feet from the fulcrum, and Jane is $6\frac{1}{2}$ feet from the fulcrum. Mary said to Jane, "I can tell how much you weigh." Can you do as well as Mary?

42. At a school entertainment the receipts from the sale of 385 tickets were $62.65. Each adult paid 25¢ and each child paid 10¢. How many adults bought tickets, and how many children?

43. The side of a square is of the same length as the side of an equilateral triangle. What is the ratio of the area of the equilateral triangle to that of the square?

44. If the side of a square is increased by 6 inches, the area of the square is doubled. How large is one side of the square?

45. The side of a square is s, and its diagonal is d. Express d in terms of s, and also s in terms of d.

46. The diagonal of one square exceeds that of another square by 3 inches. The area of the larger square exceeds that of the smaller square by 12 square inches. Find the length of each diagonal.

47. The diagonal of a rectangle is 10, and its length is 8. Find to the nearest tenth the radius of a circle having the same area as the rectangle.

48. Find two consecutive whole numbers such that $\frac{1}{7}$ of the greater exceeds $\frac{1}{9}$ of the less by 1.

49. The tens' digit of a certain number of two figures exceeds the units' digit by 3. The sum of the digits is $\frac{1}{7}$ of the number. Find the number.

50. The sum of $\frac{1}{3}$ of a number and $\frac{1}{2}$ of the same number is d. Find the number in terms of d.

51. The denominator of a fraction is double its numerator. If its numerator is increased by 3 and its denominator decreased by 4, the value of the fraction becomes 1. Find the denominator of the original fraction.

52. A certain number is doubled and the result increased by 3. When this sum is divided by 4, the quotient is 5 and the remainder is 1. What is the number?

53. Two integers are in the ratio of $3:2$. Their product exceeds their difference by 22. Find the two integers.

54. A room which was 9 feet longer than it was wide required 70 feet of molding. Find the dimensions of the room.

55. A rug is 4 feet longer than it is wide and has an area of 21 square feet. Find the dimensions of the rug.

56. The perimeter of a rectangle is 44 inches, and its area is 120 square inches. Find the dimensions of the rectangle.

57. A rectangle is m feet longer than it is wide. Its perimeter is p feet. Find its length and width in terms of m and p.

58. The side of a square is 2 inches longer than the width of a rectangle and 3 inches shorter than the length of the rectangle. The area of the rectangle is 1 square inch greater than the area of the square. Find the side of the square.

59. A city park bounded by three streets has the form of a right triangle. If the longest side of the park is 241 feet and one of the other sides is 120 feet, find the third side, and also the area of the park.

VII. Numerical Trigonometry

1. *Find the value of each of the following:*

(1) sin 15°. (3) sin 86°. (5) tan 78°. (7) tan 27° 30′.
(2) cos 32°. (4) tan 51°. (6) sin 63° 20′. (8) cos 48° 48′.

2. *Find the angle A to the nearest degree in each of the following:*

(1) $\tan A = 0.7002$. (3) $\sin A = 0.9744$. (5) $\tan A = 3.6120$.
(2) $\cos A = 0.4540$. (4) $\cos A = 0.8900$. (6) $\sin A = 0.5958$.

3. *Find the angle A to the nearest minute in each of the following:*

(1) $\sin A = 0.9832$. (3) $\cos A = 0.8936$. (5) $\tan A = 0.0600$.
(2) $\tan A = 2.5401$. (4) $\sin A = 0.2780$. (6) $\cos A = 0.3982$.

4. If the angles of a right triangle are A, B, and C, C being the right angle, and the sides opposite these angles are a, b, and c respectively, write the equation which you would use

(1) to find a, when c and angle A are known;
(2) to find b, when a and angle B are known;
(3) to find c, when a and angle B are known;
(4) to find b, when c and angle A are known;
(5) to find A, when b and a are known;
(6) to find B, when b and c are known.

5. A forest ranger desired to find the height of one of the giant redwood trees in California. He found the angle of elevation of the top of the tree, at a point 200 feet from its foot, to be 55°. How high was the tree?

6. The leaning tower of Pisa (see page 66) is inclined 5° from the vertical. How far from the building will a stone that is dropped from the first railing strike the ground, if the distance measured up the side of the building to the first railing is about 160 feet?

7. The Great Pyramid in Egypt originally was about 480 feet high and each side of its square base was approximately 755 feet long. Using these approximate values, find (1) the diagonal of its base; (2) the angle which each of the four sloping edges of the pyramid makes with a diagonal of the base.

4. A Final Test. You should now be able to perform the most important processes of elementary algebra both accurately and with reasonable speed. The following test, while not including every item mentioned in the General Summary, will show to what extent you have mastered the fundamental principles and ideas of algebra.

Mastery Test

PART I

1. In the formula $d = 100\,t + 16\,t^2$,

(1) 16 is called a ____;
(3) $100\,t$ is called a ____;
(2) 2 is called an ____;
(4) $100\,t + 16\,t^2$ is called a ____.

2. The difference between two numbers is n, and the larger number is l. What is the smaller number?

3. Write in symbols: The square of three times a number n, diminished by three times the square of the number.

4. A motorist drives a distance of d miles the first day, m miles the second day, and s miles the third day. What is the average number of miles which he drives per day?

5. If p pounds of coffee cost t dollars, how many cents per pound did the coffee cost?

6. A rug is n yards and i inches long. What is its length in feet?

7. Combine $3\,b + 7\,h - b - 7 - 3\,h + 7\,b + 4$.

8. Find the sum of $10\,a - b + 3\,c$, $7\,c - 2\,a - 5\,b$, and $a - 4\,c$. Check the result by letting $a = 2$, $b = 3$, and $c = 4$.

9. Subtract $-3\,x^2 - x - 5$ from $9\,x - 7 + x^2$. Check the result by letting $x = 3$.

10. Multiply $-8\,xy^2$ by each of the following terms:

(1) 3; (2) $-x^2$; (3) xy; (4) $-2\,y^2$; (5) $5\,x^2y$.

11. Remove parentheses and combine terms in each of the following:

(1) $-4\,a(a^2 - 3\,a + 4)$.
(3) $5 + (2\,b^2 - b) - (3\,b + 5)$.
(2) $10 - 2(3 - n) + 1$.
(4) $(2\,x - 5)^2 - (2\,x + 5)(2\,x - 5)$.

12. Multiply $3y - 2y^2 - 7$ by $5y - 1$. Check the result by letting $y = 2$.

13. Divide $12x^3y^2$ by each of the following terms:

(1) -4; (2) x; (3) $-y^2$; (4) $2x^3$; (5) $-3xy^2$.

14. Simplify $\dfrac{24m^3 - 16m^2 + 8m}{8m}$.

15. Divide $6x^3 - 19x^2 + 23x - 12$ by $2x - 3$. Check the result by letting $x = 2$.

16. Factor each of the following expressions:

(1) $3ax - 9ay + 6an^2$.
(2) $2c^2 - 9c - 18$.
(3) $a^2 + 9b^2 - 6ab$.
(4) $81 - 4n^2$.
(5) $6n^2 - 54$.
(6) $2a^2 + 16a + 32$.
(7) $a^4 - \dfrac{b^2}{16}$.
(8) $x^2y - 5xy - 14y$.

17. In the case of each of the following statements, indicate whether it is true or false. If it is false, rewrite it in correct form.

(1) $\dfrac{a}{2} \times \dfrac{2}{a} = 0$.
(2) $\dfrac{n}{d} = \dfrac{an}{ad}$.
(3) $\left(\dfrac{a+2}{b}\right)^2 = \dfrac{a^2+4}{b^2}$.
(4) $\dfrac{a}{a-b} = -\dfrac{a}{b-a}$.
(5) $\dfrac{a}{x} + \dfrac{b}{2x} = 2a + b$.
(6) $\dfrac{3x - 3y}{3x} = -3y$.

18. Reduce to lowest terms: $\dfrac{a^2 - 6a}{a^2 - 7a + 6}$.

19. Express each of the following as a single fraction in its lowest terms:

(1) $\dfrac{5x^2}{cd} \div \dfrac{10x}{c^2}$.
(2) $\dfrac{a^2 - 9}{2a + 2} \times \dfrac{2}{a + 3}$.
(3) $\dfrac{5}{4n} + \dfrac{4}{3n^2} - \dfrac{3}{8n^3}$.
(4) $\dfrac{4x + 3}{2} - \dfrac{2x - 3}{6}$.
(5) $3 - \dfrac{3r - 7}{r - 2}$.
(6) $\dfrac{3}{c^2 - d^2} - \dfrac{2}{d - c}$.

20. Simplify $\left(1 - \dfrac{x}{x-y}\right) \cdot \dfrac{x-y}{y}$.

21. Solve for n: $9n + 4 - 2n = 44 - 3n$.

22. Solve for x: $8x - 3(x - 5) = 21 + (2x - 9)$.

23. Solve for b: $\frac{3b}{4} + \frac{b}{8} = 12$.

24. Solve for x: $\frac{2}{x} - \frac{6}{x - 4} = \frac{1}{4}$.

25. If $a = 5$, find the value of $\frac{2}{3}(3a^2 + a - 7) - \left(\frac{5a}{3} - 1\right)$.

26. Solve graphically the following pair of equations:

$$3x + 4y = 26,$$
$$x - 3y = 0.$$

27. Solve algebraically the pair of equations given in Exercise 26, and check your result by substitution.

28. What are the two roots of the equation $x^2 - 6x = -5$?

29. Find the roots of the following equation, correct to the nearest tenth: $3x^2 - 5x - 4 = 0$.

30. Determine the values of x for which the following proportion is true: $3x : 5x - 4 = x - 1 : x$.

31. Find the square roots of 14 correct to the *nearest tenth*.

32. Given $\sqrt{2} = 1.414$. Find the value of $\sqrt{50}$.

33. Simplify each of the following:

(1) $\sqrt{3n} \times \sqrt{18n}$.

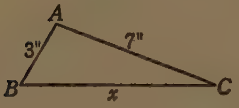
(2) $3\sqrt{\frac{1}{27}}$.

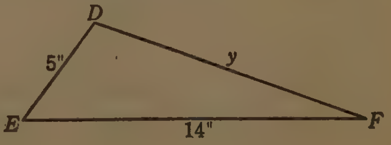
(3) $b\sqrt{27a} + 6\sqrt{3ab^2}$.

(4) $2\sqrt{\frac{5}{3}} - \sqrt{60}$.

34. In the formula $V = R - \sqrt{\frac{4R^2 - c^2}{4}}$, find the value of V when $R = 10$ and $c = 12$.

35. The triangles *ABC* and *DEF* shown below are similar. *AB* and *DE* are corresponding sides. Write a proportion involving the side *BC*, or x. How long is x? How long is y?





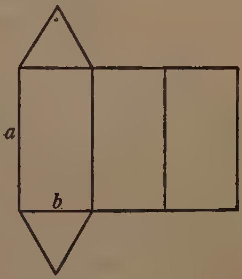
PART II

36. Write a formula for the perimeter of a rectangle whose width is w and whose length is l.

37. An automobile was worth d dollars when bought. If its value (V) decreased \$200 yearly, write a formula for determining its value after any number of years (y).

38. In an equilateral triangle the area (A) is expressed by the formula $A = \frac{b^2\sqrt{3}}{4}$, where b is the length of one side of the triangle. If $b = 4$ inches, find the area to the nearest tenth.

39. A pupil made a cardboard model of a triangular prism, by using a pattern consisting of three equal rectangles and two congruent equilateral triangles, as shown in the adjoining figure. The dimensions of each rectangle are a and b. Making use of the formula stated in Exercise 38, write a formula for the total surface (T) of the prism.



40. Solve for M the formula $L = \frac{Mt - g}{t}$.

41. It is given that $C = \frac{E}{R + \frac{r}{n}}$.

(1) What is the effect on C if R increases, while E, r, and n are constant?

(2) What is the effect on C if n increases, while E, R, and r are constant?

42. Write a formula giving the relation between f and t which is suggested by the following table:

If f is	2	3	5	7	12
then t is	6	10	18	26	46

43. A certain book company gives each new salesman a fixed salary of \$20 a week, and in addition allows him a commission of \$2 on each set of books he sells.

(1) Make a table showing what the weekly income (i) of the salesman would be if the number of sales (n) were as follows: 0, 1, 3, 5, 10, 12.

(2) Draw a graph based on this table.

(3) From the graph find the value of i when $n = 8$.

PART III

44. If 19 is subtracted from three times a certain number, the difference is 110. Find the number.

45. Two integers are in the ratio of 3 : 2. Their product exceeds their difference by 22. Find the two integers.

46. A man invested \$4500 in two enterprises, the first paying 7 % and the second 4 % annually. If his annual income from both is \$234, how much has he invested in each enterprise?

47. At a high-school baseball game children paid 35 cents admission, and adults 50 cents. If 175 people were in attendance, and the receipts were \$72.50, how many children and how many adults attended the game?

48. A dealer has two kinds of tea, worth 60¢ and 70¢ a pound respectively. How many pounds of each must be taken to make a mixture of 130 pounds worth 66¢ a pound?

49. A boat is scheduled to leave at 9:12 A.M. from a landing which is one mile from Mr. Smith's home. Suppose that Mr. Smith leaves his home at 9:00 A.M. to take the boat. If he can walk 4 miles an hour and can run 8 miles an hour, how many minutes may he walk and how many minutes must he run to reach the landing on time?

50. A man walked 1000 yards up a slope which makes an angle of 20° 15′ with the horizontal plane. How high was he then above the horizontal plane?

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